

# CENTER MANIFOLD THEOREM AND STABILITY FOR INTEGRAL EQUATIONS WITH INFINITE DELAY

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**ABSTRACT.** The present paper deals with autonomous integral equations with infinite delay via dynamical system approach. Existence, local exponential attractivity, and other properties of center manifold are established by means of the variation-of-constants formula in the phase space that is obtained in a previous paper [22]. Furthermore, we prove a stability reduction principle by which the stability of an autonomous integral equation is implied by that of an ordinary differential equation which we call the "central equation".

## 1. INTRODUCTION

In this paper we are concerned with the integral equation with infinite delay

$$(E) \quad x(t) = \int_{-\infty}^t K(t-s)x(s)ds + f(x_t),$$

where  $K$  is a measurable  $m \times m$  matrix valued function with complex components satisfying the condition

$$\int_0^\infty \|K(t)\|e^{\rho t}dt < \infty \quad \text{and} \quad \text{ess sup}\{\|K(t)\|e^{\rho t} : t \geq 0\} < \infty,$$

and  $f$  is a nonlinear term belonging to the space  $C^1(X; \mathbb{C}^m)$ , the set of all continuously (Fréchet) differentiable functions mapping  $X$  into  $\mathbb{C}^m$ , with the property that  $f(0) = 0$  and  $Df(0) = 0$ ; here,  $\rho$  is a positive constant which is fixed throughout the paper, and  $X := L_\rho^1(\mathbb{R}^-; \mathbb{C}^m)$ ,  $\mathbb{R}^- := (-\infty, 0]$ , is a Banach space which will be introduced in the next section as the phase space for Eq. (E), and  $x_t$  is an element in  $X$  defined as  $x_t(\theta) = x(t+\theta)$  for  $\theta \in \mathbb{R}^-$ . One of the purpose of the paper is to establish several results (the existence, the (local) exponential attractivity and so on) on the (local) center manifolds of the equilibrium point 0 of Eq. (E). For several kinds of equations including ordinary differential equations, functional differential equations, parabolic partial differential equations and Volterra difference equations,

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the subject has been thoroughly studied. For more information in this direction we refer the reader to the references [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 16, 17, 21, 25, 27, 28, 30, 31, 32] and the references therein. Recently, Diekmann and Gyllenberg [7] have treated Eq. (E) by Adjoint Semigroup Theory, and established several results including the principle of linearized stability for integral equations. Motivated by the pioneering paper [7], the authors in [22] have also treated integral equations with infinite delay but via a dynamical system approach, and established a "variation-of-constants formula" (VCF, for short) in the phase space. To the best of our knowledge, the existence as well as applications of invariant manifolds, in particular center manifolds for Eq. (E) are still open questions. It is the purpose of this paper to address these questions via the VCF established in [22].

Center manifolds play a crucial role in the stability analysis of systems around non-hyperbolic equilibria. The existence as well as the smoothness of center (center-stable) manifolds allow us to reduce the stability analysis of an original system to that of its restriction to a center (center stable) manifold. This procedure was initiated by Pliss [29], and subsequently, becomes popular in the mathematical literature on stability and applications. For more information on the reduction principle for ordinary differential equations in finite and infinite dimensional spaces we refer the reader to [1, 2, 4, 12, 20, 26] and the references therein. Extensions of the reduction principle to a variety of kinds of equations, including functional differential equations could be found in [1, 9, 11, 19, 20, 26, 32] and their references. Indeed, as stated in [11, Section 10.5], for any functional differential equation (FDE) with a nonhyperbolic equilibrium point 0 the stability of the FDE around 0 is reduced to that of an ordinary differential equation  $\dot{u} = h(u)$  in the neighborhood of its equilibrium point 0. The procedure can be done based on the dynamical system restricted to a center (center stable) manifold that are assumed to exist and to be sufficiently smooth.

We now outline the presentation of our paper. In Section 2 we present preliminary results necessary for our later arguments. In Subsection 3.1 of the paper, applying the VCF in [22] we will prove a center manifold theorem for Eq.(E) including the existence and the exponential attractivity (Theorem 5). In Subsection 3.2, introducing an ordinary differential equation which we call the "central equation" of Eq. (E), we will establish the reduction principle for integral equations (Theorem 6) that the stability properties for the central equation imply that of Eq. (E) in the neighborhood of its zero solution. As an application of Theorem 6 to stability analysis of some particular equations, we will consider a scalar integral equation. Indeed, by calculating the corresponding central equation we obtain a result (Proposition 9) on the stability properties for the equation in the critical case. Also, in Appendix, we give a proof of the smoothness of the center manifold. In addition, for completeness, we establish the existence of other invariant manifolds (stable manifolds and unstable manifolds etc.) for Eq. (E).

## 2. NOTATIONS AND SOME PREPARATORY RESULTS

Let  $\mathbb{N}$ ,  $\mathbb{R}^+$ ,  $\mathbb{R}^-$ ,  $\mathbb{R}$  and  $\mathbb{C}$  be the set of natural numbers, nonnegative real numbers, non-positive real numbers, real numbers and complex numbers, respectively. For an  $m \in \mathbb{N}$ , we denote by  $\mathbb{C}^m$  the space of all  $m$ -column vectors whose components are complex numbers, with the Euclidean norm  $|\cdot|$ .

Given Banach spaces  $(U, \|\cdot\|_U)$  and  $(V, \|\cdot\|_V)$ , we denote by  $\mathcal{L}(U; V)$  the space of bounded linear operators from  $U$  to  $V$  with norm

$$\|Q\|_{\mathcal{L}(U; V)} := \sup \{ \|Q(u)\|_V / \|u\|_U : u \in U, u \neq 0 \}$$

for  $Q \in \mathcal{L}(U; V)$ , and use the symbol  $\mathcal{L}(U)$  in place of  $\mathcal{L}(U; U)$ . In particular, for an  $m \times m$  matrix  $M$  with complex components,  $\|M\|$  means its operator norm  $\|M\|_{\mathcal{L}(\mathbb{C}^m)}$ .

For an interval  $J \subset \mathbb{R}$  and a Banach space  $U$  we denote by  $C(J; U)$  the space of  $U$ -valued continuous functions on  $J$ , and by  $BC(J; U)$  its subspace of bounded continuous functions on  $J$ . We also use the notation  $B_U(r)$  which stands for the open ball in  $U$  at the center 0 with radius  $r > 0$ , that is,  $B_U(r) = \{u \in U : \|u\|_U < r\}$ .

**2.1. Phase space and initial value problems.** Let  $\rho$  be a fixed positive constant, and let  $X$  be the function space  $L^1_\rho(\mathbb{R}^-; \mathbb{C}^m)$  that is defined to be all equivalent classes of measurable functions

$$\phi : \mathbb{R}^- \rightarrow \mathbb{C}^m : \phi(\theta)e^{\rho\theta} \text{ is integrable on } \mathbb{R}^-.$$

Clearly,  $X$  is a Banach space endowed with norm

$$\|\phi\|_X := \int_{-\infty}^0 |\phi(\theta)|e^{\rho\theta} d\theta, \quad \phi \in X.$$

For any function  $x : (-\infty, a) \rightarrow \mathbb{C}^m$  and  $t < a$ , we define a function  $x_t : \mathbb{R}^- \rightarrow \mathbb{C}^m$  by  $x_t(\theta) := x(t + \theta)$  for  $\theta \in \mathbb{R}^-$ ; the function  $x_t$  is called the  $t$ -segment of  $x(t)$ .

Consider the integral equations

$$(1) \quad x(t) = \int_{-\infty}^t K(t-s)x(s)ds + p(t)$$

and

$$(E) \quad x(t) = \int_{-\infty}^t K(t-s)x(s)ds + f(x_t),$$

where we assume, throughout the paper, that the kernel  $K$  is a measurable  $m \times m$  matrix valued function with complex components satisfying the conditions

$$(2) \quad \|K\|_{1,\rho} := \int_0^\infty \|K(t)\|e^{\rho t} dt < \infty, \quad \|K\|_{\infty,\rho} := \text{ess sup}\{\|K(t)\|e^{\rho t} : t \geq 0\} < \infty,$$

$p \in C(\mathbb{R}; \mathbb{C}^m)$  and  $f : X \rightarrow \mathbb{C}^m$  is of class  $C^1$ . Then Eq.(1) (resp. (E)) can be formulated as an abstract equation on the space  $X$  of the form

$$(3) \quad x(t) = F(t, x_t),$$

with  $F(t, \phi) = L(\phi) + p(t)$  (resp.  $L(\phi) + f(\phi)$ ) for  $(t, \phi) \in \mathbb{R} \times X$ , where

$$L(\phi) := \int_{-\infty}^0 K(-\theta)\phi(\theta)d\theta, \quad \phi \in X.$$

Note that, in each case,  $F(t, \phi)$  is well-defined because of

$$|L(\phi)| \leq \int_{-\infty}^0 \|K(-\theta)\| e^{-\rho\theta} |\phi(\theta)| e^{\rho\theta} d\theta \leq \|K\|_{\infty, \rho} \|\phi\|_X.$$

Thus,  $X$  may be viewed as the phase space for Eq.'s (1) and (E); in what follows we will call  $X$  the phase space.

Now let  $F : [b, \infty) \times X \rightarrow \mathbb{C}^m$  be any continuous function, and consider the equation (3) with the initial condition

$$(4) \quad x_\sigma = \phi, \quad \text{that is,} \quad x(\sigma + \theta) = \phi(\theta) \quad \text{for } \theta \in \mathbb{R}^-,$$

where  $(\sigma, \phi) \in [b, \infty) \times X$  is given arbitrarily. A function  $x : (-\infty, a) \rightarrow \mathbb{C}^m$  is said to be a solution of the initial value problem (3)-(4) on the interval  $(\sigma, a)$  if  $x$  satisfies the following conditions:

- (i)  $x_\sigma = \phi$ , that is,  $x(\sigma + \theta) = \phi(\theta)$  for  $\theta \in \mathbb{R}^-$ ;
- (ii)  $x \in L^1_{\text{loc}}[\sigma, a)$ ,  $x$  is locally integrable on  $[\sigma, a)$ ;
- (iii)  $x(t) = F(t, x_t)$  for  $t \in (\sigma, a)$ .

If  $F(t, \phi)$  is locally Lipschitz continuous in  $\phi$ , by [22, Proposition 1] the initial value problem (3)-(4) has a unique (local) solution, which is defined globally if, in particular,  $F(t, \phi)$  is globally Lipschitz continuous in  $\phi$  ([22, Proposition 3]). So for any  $(\sigma, \phi) \in \mathbb{R} \times X$  (1)-(4) has a unique global solution, denoted  $x(t; \sigma, \phi, p)$ , which is called the solution of Eq.(1) through  $(\sigma, \phi)$ . Similarly, (E)-(4) has a unique (local) solution, which is denoted by  $x(t; \sigma, \phi, f)$ . Moreover we remark that if  $x(t)$  is a solution of Eq.(3) on  $(\sigma, a)$ , then  $x_t$  is an  $X$ -valued continuous function on  $[\sigma, a)$  (see [22, Lemma 1]).

Now suppose that  $\phi = \psi$  in  $X$ , that is,  $\phi(\theta) = \psi(\theta)$  a.e.  $\theta \in \mathbb{R}^-$ . Then by the uniqueness of solutions of (1)-(4) it follows that  $x(t; \sigma, \phi, p) = x(t; \sigma, \psi, p)$  for  $t \in (\sigma, \infty)$ , so that  $x_t(\sigma, \phi, p) = x_t(\sigma, \psi, p)$  in  $X$  for  $t \in [\sigma, \infty)$ . In particular, given  $\sigma \in \mathbb{R}$ ,  $x_t(\sigma, \cdot, p)$  induces a transformation on  $X$  for each  $t \in [\sigma, \infty)$ ; and similarly for  $x_t(\sigma, \cdot, f)$  with  $t \in [\sigma, a)$  provided that  $x(t; \sigma, \phi, f)$  is the solution of (E)-(4) on  $(\sigma, a)$ .

When  $J$  is an interval in  $\mathbb{R}$ , a function  $\xi(t)$  is called a solution of Eq. (1) on  $J$ , if  $\xi_t \in X$  is defined for all  $t \in J$  and if it satisfies  $x(t; \sigma, \xi_\sigma, p) = \xi(t)$  for all  $t$  and  $\sigma$  in  $J$  with  $t \geq \sigma$ ; and, similarly, a function  $\xi(t)$  is called a solution of Eq. (E) on  $J$  whenever  $\xi_t \in X$  for  $t \in J$ , and  $x(t; \sigma, \xi_\sigma, f) = \xi(t)$  holds for all  $t$  and  $\sigma$  in  $J$  with  $t \geq \sigma$ .

**2.2. A Variation-of-constants formula and decomposition of the phase space.** Now, for any  $t \geq 0$  and  $\phi \in X$ , we define  $T(t)\phi \in X$  by

$$\begin{aligned} [T(t)\phi](\theta) &:= x_t(\theta; 0, \phi, 0) \\ &= \begin{cases} x(t + \theta; 0, \phi, 0), & -t < \theta \leq 0, \\ \phi(t + \theta), & \theta \leq -t. \end{cases} \end{aligned}$$

Then  $T(t)$  defines a bounded linear operator on  $X$ . We call  $T(t)$  the solution operator of the homogeneous integral equation

$$(5) \quad x(t) = \int_{-\infty}^t K(t-s)x(s)ds.$$

$\{T(t)\}_{t \geq 0}$  is a strongly continuous semigroup of bounded linear operators on  $X$ , called a solution semigroup for Eq.(5).

Given a positive integer  $n$ , we introduce a continuous function  $\Gamma^n : \mathbb{R}^- \rightarrow \mathbb{R}^+$  which is of compact support with  $\text{supp } \Gamma^n \subset [-1/n, 0]$  and satisfies  $\int_{-\infty}^0 \Gamma^n(\theta)d\theta = 1$ . Obviously,  $\Gamma^n x \in X$  for  $x \in \mathbb{C}^m$  and the inequality  $\|\Gamma^n x\|_X \leq |x|$  holds.

The following theorem, established in [22], gives a representation formula for solutions of Eq. (1) in the phase space  $X$ , which is called the variation-of-constants formula (VCF, for short) in the phase space and plays an essential role in the present paper.

**Theorem 1.** [22, Theorem 3] *The segment  $x_t(\sigma, \phi, p)$  of the solution  $x(\cdot; \sigma, \phi, p)$  of Eq.(1) satisfies the following relation in  $X$ :*

$$x_t(\sigma, \phi, p) = T(t - \sigma)\phi + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t-s)(\Gamma^n p(s))ds, \quad t \geq \sigma.$$

Let  $\overline{X}$  be a subset of  $X$  of elements  $\phi \in X$  which are continuous on  $[-\varepsilon_\phi, 0]$  for some  $\varepsilon_\phi > 0$ , and set

$$X_0 := \{\psi \in X : \psi = \phi \text{ a.e. on } \mathbb{R}^- \text{ for some } \phi \in \overline{X}\}.$$

Then for any  $\psi \in X_0$  we can define the value of  $\psi$  at  $\theta = 0$  by

$$\psi[0] := \phi(0),$$

where  $\phi$  is an element in  $\overline{X}$  satisfying  $\psi = \phi$  a.e. on  $\mathbb{R}^-$ . It is clear that  $\psi[0]$  is well-defined, and  $X_0$  is a normed space equipped with norm

$$\|\psi\|_{X_0} := \|\psi\|_X + |\psi[0]|, \quad \psi \in X_0.$$

By [22, Lemma 1], we note that the solution  $x(\cdot; \sigma, \phi, p)$  of Eq. (1) through  $(\sigma, \phi) \in \mathbb{R} \times X$  satisfies  $x_t(\sigma, \phi, p) \in X_0$  with  $(x_t(\sigma, \phi, p))[0] = x(t; \sigma, \phi, p)$  for  $t > \sigma$ .

The following result yields an intimate relation between solutions of Eq. (1) and  $X$ -valued functions satisfying an integral equation which arises from the variation-of-constants formula in the phase space.

**Theorem 2.** [22, Theorem 4] *Let  $p \in C(\mathbb{R}; \mathbb{C}^m)$ .*

(i) *If  $x(t)$  is a solution of Eq. (1) on the entire  $\mathbb{R}$ , then the  $X$ -valued function  $\xi(t) := x_t$  satisfies the relations*

$$(a) \quad \xi(t) = T(t - \sigma)\xi(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t - s)(\Gamma^n p(s))ds, \quad \forall (t, \sigma) \in \mathbb{R}^2 \text{ with } t \geq \sigma, \text{ in } X;$$

$$(b) \quad \xi \in C(\mathbb{R}; X_0).$$

(ii) *Conversely, if a function  $\xi : \mathbb{R} \rightarrow X$  satisfies the relation*

$$\xi(t) = T(t - \sigma)\xi(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t - s)(\Gamma^n p(s))ds, \quad \forall (t, \sigma) \in \mathbb{R}^2 \text{ with } t \geq \sigma,$$

*then*

$$(c) \quad \xi \in C(\mathbb{R}; X_0);$$

(d) *if we set*

$$u(t) = (\xi(t))[0], \quad \forall t \in \mathbb{R},$$

*then  $u \in C(\mathbb{R}; \mathbb{C}^m)$ ,  $u_t = \xi(t)$  (in  $X$ ) for any  $t \in \mathbb{R}$  and  $u$  is a solution of Eq. (1) on  $\mathbb{R}$ .*

Based on spectral analysis of the generator  $A$  of the solution semigroup  $\{T(t)\}_{t \geq 0}$ , we also have established the decomposition theorem of the phase space  $X$  ([22]): Let  $\sigma(A)$  and  $P_{\sigma}(A)$  be the spectrum and the point spectrum of the generator  $A$ , respectively. Then the following relation holds between the spectrum of  $A$  and the characteristic roots of Eq. (5)

$$\sigma(A) \cap \mathbb{C}_{-\rho} = P_{\sigma}(A) \cap \mathbb{C}_{-\rho} = \{\lambda \in \mathbb{C}_{-\rho} : \det \Delta(\lambda) = 0\},$$

where  $\mathbb{C}_{-\rho} := \{z \in \mathbb{C} : \operatorname{Re} z > -\rho\}$ , and  $\Delta(\lambda)$  is the characteristic operator of Eq. (5), that is,

$$\Delta(\lambda) := E_m - \int_0^{\infty} K(t)e^{-\lambda t}dt,$$

$E_m$  being the  $m \times m$ -unit matrix ([22, Proposition 4]). Moreover, for  $\operatorname{ess}(A)$ , the essential spectrum of  $A$ , we have the estimate  $\sup_{\lambda \in \operatorname{ess}(A)} \operatorname{Re} \lambda \leq -\rho$  ([22, Corollary 2]). Now set  $\Sigma^u := \{\lambda \in \sigma(A) : \operatorname{Re} \lambda > 0\}$ ,  $\Sigma^c := \{\lambda \in \sigma(A) : \operatorname{Re} \lambda = 0\}$ , and  $\Sigma^s := \sigma(A) \setminus (\Sigma^c \cup \Sigma^u)$ . Then these observations, combined with the analyticity of  $\det \Delta(\lambda)$  on the domain  $\mathbb{C}_{-\rho}$ , yield the following result.

**Theorem 3.** [22, Theorem 2] *Let  $\{T(t)\}_{t \geq 0}$  be the solution semigroup of Eq. (5). Then  $X$  is decomposed as a direct sum of closed subspaces  $E^u$ ,  $E^c$ , and  $E^s$*

$$X = E^u \oplus E^c \oplus E^s$$

*with the following properties:*

$$(i) \quad \dim(E^u \oplus E^c) < \infty,$$

$$(ii) \quad T(t)E^u \subset E^u, T(t)E^c \subset E^c, \text{ and } T(t)E^s \subset E^s \text{ for } t \in \mathbb{R}^+,$$

- (iii)  $\sigma(A|_{E^u}) = \Sigma^u$ ,  $\sigma(A|_{E^c}) = \Sigma^c$  and  $\sigma(A|_{E^s \cap \mathcal{D}(A)}) = \Sigma^s$ ,
- (iv)  $T^u(t) := T(t)|_{E^u}$  and  $T^c(t) := T(t)|_{E^c}$  are extendable for  $t \in \mathbb{R}$  as groups of bounded linear operators on  $E^u$  and  $E^c$ , respectively,
- (v)  $T^s(t) := T(t)|_{E^s}$  is a strongly continuous semigroup of bounded linear operators on  $E^s$ , and its generator is identical with  $A|_{E^s \cap \mathcal{D}(A)}$ ,
- (vi) there exist positive constants  $\alpha, \varepsilon$  with  $\alpha > \varepsilon$  and a constant  $C \geq 1$  such that

$$\begin{aligned} \|T^s(t)\|_{\mathcal{L}(X)} &\leq Ce^{-\alpha t}, \quad t \in \mathbb{R}^+, \\ \|T^u(t)\|_{\mathcal{L}(X)} &\leq Ce^{\alpha t}, \quad t \in \mathbb{R}^-, \\ \|T^c(t)\|_{\mathcal{L}(X)} &\leq Ce^{\varepsilon|t|}, \quad t \in \mathbb{R}. \end{aligned}$$

In (vi) we note that  $C$  is a constant depending only on  $\alpha$  and  $\varepsilon$ , and that the value of  $\varepsilon > 0$  can be taken arbitrarily small. Also, we will use the notations  $E^{cu} = E^c \oplus E^u$ ,  $E^{su} = E^s \oplus E^u$  etc, and denote by  $\Pi^s$  the projection from  $X$  onto  $E^s$  along  $E^{cu}$ , and similarly for  $\Pi^u$ ,  $\Pi^{cu}$  etc. In addition, we set

$$C_1 := \|\Pi^s\|_{\mathcal{L}(X)} + \|\Pi^c\|_{\mathcal{L}(X)} + \|\Pi^u\|_{\mathcal{L}(X)}.$$

If  $f \in C^1(X; \mathbb{C}^m)$  satisfies  $f(0) = 0$  and  $Df(0) = 0$ , Eq. (5) is the linearized equation of Eq.(E) around the equilibrium point 0. The equilibrium point 0 (or the zero solution) of Eq. (E) is said to be *hyperbolic* provided that  $\Delta(\lambda)$  is invertible on the imaginary axis; in other words,  $\Sigma^c = \emptyset$ .

### 3. CENTER MANIFOLD THEOREM FOR INTEGRAL EQUATIONS

In what follows we assume that  $f \in C^1(X; \mathbb{C}^m)$  satisfies  $f(0) = 0$  and  $Df(0) = 0$ . In this section we will establish the existence of local center manifolds of the equilibrium point 0 of Eq.(E) and study their properties. To do so, in parallel with Eq.(E), we will consider a modified equation of (E) of the form

$$(E_\delta) \quad x(t) = \int_{-\infty}^t K(t-s)x(s)ds + f_\delta(x_t),$$

where  $f_\delta$  with  $\delta > 0$  is a modification of the original nonlinear term  $f$ ; more precisely let  $\chi : \mathbb{R} \rightarrow [0, 1]$  be a  $C^\infty$ -function such that  $\chi(t) = 1$  ( $|t| \leq 2$ ) and  $\chi(t) = 0$  ( $|t| \geq 3$ ), and define

$$f_\delta(\phi) := \chi(\|\Pi^{su}\phi\|_X/\delta)\chi(\|\Pi^c\phi\|_X/\delta)f(\phi), \quad \phi \in X.$$

The function  $f_\delta : X \rightarrow \mathbb{C}^m$  is continuous on  $X$ , and is of class  $C^1$  when restricted to the open set  $S_\delta := \{\phi \in X : \|\Pi^{su}\phi\|_X < \delta\}$  since we may assume that  $\|\Pi^c\phi\|_X$  is of class  $C^1$  for  $\phi \neq 0$  because of  $\dim E^c < \infty$ . Moreover, by the assumption  $f(0) = Df(0) = 0$ , there exist a  $\delta_1 > 0$  and a nondecreasing continuous function  $\zeta_* : (0, \delta_1] \rightarrow \mathbb{R}^+$  such that  $\zeta_*(+0) = 0$ ,

$$(6) \quad \|f_\delta(\phi)\|_X \leq \delta\zeta_*(\delta) \quad \text{and} \quad \|f_\delta(\phi) - f_\delta(\psi)\|_X \leq \zeta_*(\delta)\|\phi - \psi\|_X$$

for  $\phi, \psi \in X$  and  $\delta \in (0, \delta_1]$ . Indeed, we may put

$$\zeta_*(\delta) = \left( \sup_{\|\phi\|_X \leq 3\delta} \|Df(\phi)\|_{\mathcal{L}(X; \mathbb{C}^m)} \right) \cdot \left( 1 + 3 \sup_{0 \leq t \leq 3} |\chi'(t)| \right)$$

(cf. [6, Lemma 4.1]). Taking  $\delta_1 > 0$  small, we may also assume that there exists a positive number  $M_1(\delta_1) =: M_1$  such that

$$(7) \quad \|Df_\delta(\phi)\|_{\mathcal{L}(X; \mathbb{C}^m)} \leq M_1, \quad \phi \in S_\delta$$

for any  $\delta \in (0, \delta_1]$ . Fix a positive number  $\eta$  such that

$$\varepsilon < \eta < \alpha,$$

where  $\varepsilon$  and  $\alpha$  are the constants in Theorem 3.

**3.1. Center Manifold and its exponential attractivity.** For the existence of center manifold for  $\text{Eq.}(E_\delta)$  and its exponential attractivity, we have the following:

**Theorem 4.** *There exist a positive number  $\delta$  and a  $C^1$ -map  $F_{*,\delta} : E^c \rightarrow E^{su}$  with  $F_{*,\delta}(0) = 0$  such that the following properties hold:*

- (i)  $W_\delta^c := \text{graph } F_{*,\delta}$  is tangent to  $E^c$  at zero,
- (ii)  $W_\delta^c$  is invariant for  $\text{Eq.}(E_\delta)$ , that is, if  $\xi \in W_\delta^c$ , then  $x_t(0, \xi, f) \in W_\delta^c$  for  $t \in \mathbb{R}$ .
- (iii) Assume moreover that  $\Sigma^u = \emptyset$ . Then there exists a positive constant  $\beta_0$  with the property that if  $x$  is a solution of  $\text{Eq.}(E_\delta)$  on an interval  $J = [t_0, t_1]$ , then the inequality

$$\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \leq C \|\Pi^s x_{t_0} - F_{*,\delta}(\Pi^c x_{t_0})\|_X e^{-\beta_0(t-t_0)}, \quad t \in J$$

holds true. In particular, if  $x$  is a solution on an interval  $[t_0, \infty)$ ,  $x_t$  tends to  $W_\delta^c$  exponentially as  $t \rightarrow \infty$ .

As will be shown in Proposition 3 given later, the map  $F_{*,\delta} : E^c \rightarrow E^{su}$  in the above theorem is globally Lipschitz continuous with the Lipschitz constant  $L(\delta) = 4C^2 C_1 \zeta_*(\delta) / (\alpha - \eta)$ . Noticing that  $L(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ , one can assume that the number  $\delta$  satisfies  $\delta \in (0, \delta_1]$  together with  $L(\delta) \leq 1$ . Let us take a small  $r \in (0, \delta)$  so that  $\|F_{*,\delta}(\psi)\|_X < \delta$  for any  $\psi \in B_{E^c}(r)$ . Such a choice of  $r$  is possible by the continuity of  $F_{*,\delta}$ . Set  $F_* := F_{*,\delta}|_{B_{E^c}(r)}$  and consider an open neighborhood  $\Omega_0$  of 0 in  $X$  defined by

$$\Omega_0 := \{\phi \in X : \|\Pi^{su} \phi\|_X < \delta, \|\Pi^c \phi\|_X < r\}.$$

Observe that  $f \equiv f_\delta$  on  $\Omega_0$ . Then the following theorem which yields a local center manifold for  $\text{Eq.}(E)$  as the graph of  $F_*$  immediately follows from Theorem 4.

**Theorem 5.** *Assume that  $f \in C^1(X; \mathbb{C}^m)$  with  $f(0) = Df(0) = 0$ . Then there exist positive numbers  $r, \delta$ , and a  $C^1$ -map  $F_* : B_{E^c}(r) \rightarrow E^{su}$  with  $F_*(0) = 0$ , together with an open neighborhood  $\Omega_0$  of 0 in  $X$ , such that the following properties hold:*



- (i)  $W_{\text{loc}}^c(r, \delta) := \text{graph } F_*$  is tangent to  $E^c$  at zero,
- (ii)  $W_{\text{loc}}^c(r, \delta)$  is locally invariant for Eq. (E), that is,
  - (a) for any  $\xi \in W_{\text{loc}}^c(r, \delta)$  there exists a  $t_\xi > 0$  such that  $x_t(0, \xi, f) \in W_{\text{loc}}^c(r, \delta)$  for  $|t| \leq t_\xi$ ,
  - (b) if  $\xi \in W_{\text{loc}}^c(r, \delta)$  and  $x_t(0, \xi, f) \in \Omega_0$  for  $0 \leq t \leq T$ , then  $x_t(0, \xi, f) \in W_{\text{loc}}^c(r, \delta)$  for  $0 \leq t \leq T$ .
- (iii) Assume moreover that  $\Sigma^u = \emptyset$ . Then there exists a positive constant  $\beta_0$  with the property that if  $x$  is a solution of Eq. (E) on an interval  $J = [t_0, t_1]$  satisfying  $x_t \in \Omega_0$  on  $J$ , then the inequality

$$\|\Pi^s x_t - F_*(\Pi^c x_t)\|_X \leq C \|\Pi^s x_{t_0} - F_*(\Pi^c x_{t_0})\|_X e^{-\beta_0(t-t_0)}, \quad t \in J$$

holds true. In particular, if the solution  $x(t)$  is defined on  $[t_0, \infty)$  satisfying  $x_t \in \Omega_0$  on  $[t_0, \infty)$ , then  $x_t$  tends to  $W_{\text{loc}}^c(r, \delta)$  exponentially as  $t \rightarrow \infty$ .

In what follows we will prove Theorem 4 by establishing several propositions. Before doing so, we prepare the following lemma:

**Lemma 1.** Let  $f_* \in C(X; \mathbb{C}^m)$ , and consider the equation

$$(E_*) \quad x(t) = \int_{-\infty}^t K(t-s)x(s)ds + f_*(x_t).$$

Moreover, let  $\psi \in E^c$ , and  $\eta$  be as above. Then we have:

- (i) If  $x(t)$  is a solution of Eq.  $(E_*)$  defined on  $\mathbb{R}$  with the properties that  $\Pi^c x_0 = \psi$ ,  $\sup_{t \in \mathbb{R}} \|x_t\|_X e^{-\eta|t|} < \infty$  and  $\sup_{t \in \mathbb{R}} |f_*(x_t)| < \infty$ , then the  $X$ -valued function  $u(t) := x_t$  satisfies

$$\begin{aligned} u(t) &= T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s)\Pi^c \Gamma^n f_*(u(s))ds \\ &\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s)\Pi^u \Gamma^n f_*(u(s))ds + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s)\Pi^s \Gamma^n f_*(u(s))ds \end{aligned}$$

for  $t \in \mathbb{R}$ , and moreover  $u$  belongs to  $C(\mathbb{R}; X_0)$ .

- (ii) Conversely, if  $y \in C(\mathbb{R}; X)$  with  $\sup_{t \in \mathbb{R}} \|y(t)\|_X e^{-\eta|t|} < \infty$  and  $\sup_{t \in \mathbb{R}} |f_*(y(t))| < \infty$  satisfies

$$\begin{aligned} y(t) &= T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s)\Pi^c \Gamma^n f_*(y(s))d\tau \\ &\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s)\Pi^u \Gamma^n f_*(y(s))ds + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s)\Pi^s \Gamma^n f_*(y(s))ds \end{aligned}$$

for  $t \in \mathbb{R}$ , then  $y$  belongs to  $C(\mathbb{R}; X_0)$  and the function  $\xi(t)$  defined by

$$\xi(t) := (y(t))[0], \quad t \in \mathbb{R}$$

is a solution of Eq.  $(E_*)$  on  $\mathbb{R}$  satisfying  $\Pi^c \xi_0 = \psi$ ,  $\sup_{t \in \mathbb{R}} \|\xi_t\|_X e^{-\eta|t|} < \infty$  and  $\xi_t = y(t)$  for  $t \in \mathbb{R}$ .

*Proof.* (i) Let  $p(t) = f_*(u(t))$  for  $t \in \mathbb{R}$ . Then  $p$  belongs to  $BC(\mathbb{R}; \mathbb{C}^m)$  and  $x$  is a solution of Eq. (1) on  $\mathbb{R}$ . So it follows from Theorem 2 that  $u(t) = x_t$  is a continuous  $X_0$ -valued function for  $t \in \mathbb{R}$ , that is,  $u \in C(\mathbb{R}; X_0)$ . We know from Theorem 1 that

$$(8) \quad u(t) = T(t - \sigma)u(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T(t - s) \Gamma^n p(s) ds$$

holds for any  $t$  and  $\sigma$  with  $t \geq \sigma$ , which implies

$$\Pi^{cu} u(t) = T^{cu}(t - \sigma) \Pi^{cu} u(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T^{cu}(t - s) \Pi^{cu} \Gamma^n p(s) ds,$$

and moreover

$$\begin{aligned} \Pi^{cu} u(\sigma) &= T^{cu}(\sigma - t) \left[ \Pi^{cu} u(t) - \lim_{n \rightarrow \infty} \int_{\sigma}^t T^{cu}(t - s) \Pi^{cu} \Gamma^n p(s) ds \right] \\ &= T^{cu}(\sigma - t) \Pi^{cu} u(t) - \lim_{n \rightarrow \infty} \int_{\sigma}^t T^{cu}(\sigma - s) \Pi^{cu} \Gamma^n p(s) ds, \quad t \geq \sigma \end{aligned}$$

because of the group property of  $\{T^{cu}(t)\}_{t \in \mathbb{R}}$ . So it follows that

$$\Pi^{cu} u(t) = T^{cu}(t - \sigma) \Pi^{cu} u(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T^{cu}(t - s) \Pi^{cu} \Gamma^n p(s) ds$$

for any  $t, \sigma \in \mathbb{R}$ , and in particular

$$(9) \quad \Pi^u u(\sigma) = T^u(\sigma - t) \Pi^u u(t) - \lim_{n \rightarrow \infty} \int_{\sigma}^t T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds, \quad \forall t, \sigma \in \mathbb{R}$$

and

$$(10) \quad \Pi^c u(t) = T^c(t) \psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t - s) \Pi^c \Gamma^n p(s) ds, \quad t \in \mathbb{R}.$$

Now, in view of

$$\|T^u(\sigma - t) \Pi^u u(t)\|_X \leq CC_1 e^{\alpha(\sigma - t)} e^{\eta t} \left( \sup_{t \in \mathbb{R}} \|x_t\|_X e^{-\eta t} \right) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

we get from (9)

$$\Pi^u u(\sigma) = - \lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\sigma}^t T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds, \quad \sigma \in \mathbb{R}.$$

Note that the limit

$$\lim_{t \rightarrow \infty} \int_{\sigma}^t T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds = \int_{\sigma}^{\infty} T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds$$

exists in  $X$ . Indeed, using the inequality  $\|\Gamma^n x\|_X \leq |x|$  for  $x \in \mathbb{C}^m$ , we have for  $t_2 \geq t_1 \geq \sigma$

$$\left\| \int_{t_1}^{t_2} T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds \right\|_X \leq \frac{CC_1}{\alpha} \left( \sup_{t \in \mathbb{R}} |p(t)| \right) e^{\alpha(\sigma - t_1)} \rightarrow 0$$

as  $t_1 \rightarrow \infty$ , which implies the existence of the limit, together with uniformity in  $n$  of the convergence. On the other hand, since

$$\begin{aligned}
 & \left\| \int_{\sigma}^{\infty} T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds - \int_{\sigma}^{\infty} T^u(\sigma - s) \Pi^u \Gamma^m p(s) ds \right\|_X \\
 & \leq \left\| \int_t^{\infty} T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds \right\|_X + \left\| \int_t^{\infty} T^u(\sigma - s) \Pi^u \Gamma^m p(s) ds \right\|_X \\
 & \quad + \left\| \int_{\sigma}^t T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds - \int_{\sigma}^t T^u(\sigma - s) \Pi^u \Gamma^m p(s) ds \right\|_X \\
 & \leq \frac{2CC_1}{\alpha} \left( \sup_{t \in \mathbb{R}} |p(t)| \right) e^{\alpha(\sigma-t)} \\
 & \quad + \left\| \Pi^u \left( \int_{\sigma}^t T(\sigma - s) \Gamma^n p(s) ds - \int_{\sigma}^t T(\sigma - s) \Gamma^m p(s) ds \right) \right\|_X,
 \end{aligned}$$

for any  $t > \sigma$ , it follows from  $\lim_{n \rightarrow \infty} \int_{\sigma}^t T(t - s) \Gamma^n p(s) ds = x_t(\sigma, 0, p)$  (Theorem 1) that

$$\begin{aligned}
 & \limsup_{n, m \rightarrow \infty} \left\| \int_{\sigma}^{\infty} T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds - \int_{\sigma}^{\infty} T^u(\sigma - s) \Pi^u \Gamma^m p(s) ds \right\|_X \\
 & \leq \frac{2CC_1}{\alpha} \left( \sup_{t \in \mathbb{R}} |p(t)| \right) e^{\alpha(\sigma-t)}.
 \end{aligned}$$

Since  $t > \sigma$  is arbitrary,  $\int_{\sigma}^{\infty} T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds$  converges in  $X$  as  $n \rightarrow \infty$ . These observations yield

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \int_{\sigma}^{\infty} T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds &= \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \int_{\sigma}^t T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds \\
 &= \lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\sigma}^t T^u(\sigma - s) \Pi^u \Gamma^n p(s) ds \\
 &= -\Pi^u u(\sigma), \quad \sigma \in \mathbb{R}.
 \end{aligned} \tag{11}$$

Similarly since (8) also implies

$$\Pi^s u(t) = T^s(t - \sigma) \Pi^s u(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t T^s(t - s) \Pi^s \Gamma^n p(s) ds, \quad t \geq \sigma,$$

by the same reasoning as above, one can obtain

$$\Pi^s u(t) = \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t - s) \Pi^s \Gamma^n p(s) ds, \quad t \in \mathbb{R}. \tag{12}$$

Thus, (10), (11) and (12) yield

$$\begin{aligned} u(t) &= \Pi^c u(t) + \Pi^u u(t) + \Pi^s u(t) \\ &= T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s) \Pi^c \Gamma^n p(s) ds \\ &\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s) \Pi^u \Gamma^n p(s) ds + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s) \Pi^s \Gamma^n p(s) ds, \end{aligned}$$

$t \in \mathbb{R}$ , as required.

(ii) Set  $g(t) = f_*(y(t))$  for  $t \in \mathbb{R}$ . Then  $g$  belongs to  $BC(\mathbb{R}; \mathbb{C}^m)$ , and by the same argument as the one in (i), the limits  $\lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s) \Pi^u \Gamma^n g(s) ds$  and  $\lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s) \Pi^s \Gamma^n g(s) ds$  exist for each  $t \in \mathbb{R}$ . For any  $(t, \sigma) \in \mathbb{R}^2$  with  $t \geq \sigma$ , we get the relation

$$y(t) = T(t-\sigma)y(\sigma) + \lim_{n \rightarrow \infty} \int_\sigma^t T(t-s) \Gamma^n g(s) ds$$

in  $X$ , because

$$\begin{aligned} &T(t-\sigma)y(\sigma) + \lim_{n \rightarrow \infty} \int_\sigma^t T(t-s) \Gamma^n g(s) ds \\ &= T(t-\sigma) \left\{ T^c(\sigma)\psi + \lim_{n \rightarrow \infty} \int_0^\sigma T^c(\sigma-s) \Pi^c \Gamma^n g(s) ds \right. \\ &\quad \left. - \lim_{n \rightarrow \infty} \int_\sigma^\infty T^u(\sigma-s) \Pi^u \Gamma^n g(s) ds + \lim_{n \rightarrow \infty} \int_{-\infty}^\sigma T^s(\sigma-s) \Pi^s \Gamma^n g(s) ds \right\} \\ &\quad + \lim_{n \rightarrow \infty} \int_\sigma^t T(t-s) \Gamma^n g(s) ds \\ &= T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^\sigma T^c(t-s) \Pi^c \Gamma^n g(s) ds - \lim_{n \rightarrow \infty} \int_\sigma^\infty T^u(t-s) \Pi^u \Gamma^n g(s) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^\sigma T^s(t-s) \Pi^s \Gamma^n g(s) ds + \lim_{n \rightarrow \infty} \int_\sigma^t T(t-s) \Gamma^n g(s) ds \\ &= T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s) \Pi^c \Gamma^n g(s) ds - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s) \Pi^u \Gamma^n g(s) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s) \Pi^s \Gamma^n g(s) ds \\ &= y(t) \end{aligned}$$

in  $X$ . Therefore, from Theorem 2 it follows that  $y \in C(\mathbb{R}; X_0)$  and the function  $\xi$  defined by  $\xi(t) = (y(t))[0]$ ,  $t \in \mathbb{R}$ , satisfies  $\xi \in C(\mathbb{R}; \mathbb{C}^m)$ ,  $\xi_t = y(t)$  (in  $X$ ) for  $t \in \mathbb{R}$  and  $\xi$  is a solution of Eq. (1) (with  $p = g$ ) on  $\mathbb{R}$ . Observe that  $\Pi^c \xi_0 = \Pi^c \psi = \psi$  and  $\sup_{t \in \mathbb{R}} \|\xi_t\|_X e^{-\eta|t|} = \sup_{t \in \mathbb{R}} \|y(t)\|_X e^{-\eta|t|} < \infty$ . Also, since  $f_*(\xi_t) = f_*(y(t)) = g(t)$  on  $\mathbb{R}$ ,  $\xi$  must be a solution of Eq.  $(E_*)$  on  $\mathbb{R}$ . Thus  $\xi$  is a solution of Eq.  $(E_*)$  with the desired properties. The proof is completed.  $\square$

Now take a  $\delta_1 > 0$  sufficiently small so that

$$(13) \quad \zeta_*(\delta_1)CC_1 \left( \frac{1}{\eta - \varepsilon} + \frac{2}{\alpha + \eta} + \frac{2}{\alpha - \eta} \right) < \frac{1}{2}$$

holds. Let  $Y_\eta$  be the Banach space

$$Y_\eta := \left\{ y \in C(\mathbb{R}; X) : \sup_{t \in \mathbb{R}} \|y(t)\|_X e^{-\eta|t|} < \infty \right\}$$

with norm

$$\|y\|_{Y_\eta} := \sup_{t \in \mathbb{R}} \|y(t)\|_X e^{-\eta|t|}, \quad y \in Y_\eta,$$

and for  $(\psi, y) \in E^c \times Y_\eta$  set

$$(14) \quad \begin{aligned} \mathcal{F}_\delta(\psi, y)(t) &:= T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s) \Pi^c \Gamma^n f_\delta(y(s)) ds \\ &\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s) \Pi^u \Gamma^n f_\delta(y(s)) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s) \Pi^s \Gamma^n f_\delta(y(s)) ds \end{aligned}$$

for  $t \in \mathbb{R}$ . Notice that the right-hand side is well-defined and that  $\mathcal{F}_\delta(\psi, y)$  is an  $X$ -valued function on  $\mathbb{R}$  for each  $(\psi, y) \in E^c \times Y_\eta$ .

In the following we will establish several propositions to prove Theorem 4.

**Proposition 1.** *Let  $\mathcal{F}_\delta(\psi, y)$  be as above. Then*

- (i)  $\mathcal{F}_\delta$  defines a map from  $E^c \times Y_\eta$  to  $Y_\eta$  by sending  $(\psi, y) \in E^c \times Y_\eta$  to  $\mathcal{F}_\delta(\psi, y)$ .
- (ii) Let  $\delta \in (0, \delta_1]$ . Then  $\mathcal{F}_\delta(\psi, \cdot)$  is a contraction map from  $Y_\eta$  into itself, with Lipschitz constant  $1/2$ , for each  $\psi \in E^c$ .

*Proof.* (i) We first show  $\mathcal{F}_\delta(\psi, y) \in C(\mathbb{R}; X)$ . Let  $z(t) := \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s) \Pi^s \Gamma^n f_\delta(y(s)) ds = \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s) \Pi^s \Gamma^n p_\delta(s) ds$  for  $t \in \mathbb{R}$ , where  $p_\delta(t) := f_\delta(y(t))$ . Take a  $\sigma$  so that  $\sigma < t$ . Then

$$\begin{aligned} z(t) &= \lim_{n \rightarrow \infty} \int_{-\infty}^\sigma T^s(t-s) \Pi^s \Gamma^n p_\delta(s) ds + \lim_{n \rightarrow \infty} \int_\sigma^t T^s(t-s) \Pi^s \Gamma^n p_\delta(s) ds \\ &= T^s(t-\sigma) \left( \lim_{n \rightarrow \infty} \int_{-\infty}^\sigma T^s(\sigma-s) \Pi^s \Gamma^n p_\delta(s) ds \right) + \Pi^s \left( \lim_{n \rightarrow \infty} \int_\sigma^t T(t-s) \Gamma^n p_\delta(s) ds \right) \\ &= \Pi^s \left\{ T(t-\sigma)z(\sigma) + \lim_{n \rightarrow \infty} \int_\sigma^t T(t-s) \Gamma^n p_\delta(s) ds \right\}. \end{aligned}$$

Observe that the  $X$ -valued function  $T(t-\sigma)z(\sigma)$  is continuous in  $t$ . Also, the term  $\lim_{n \rightarrow \infty} \int_\sigma^t T(t-s) \Gamma^n p_\delta(s) ds$  is continuous in  $t$  as an  $X$ -valued function, because of  $\lim_{n \rightarrow \infty} \int_\sigma^t T(t-s) \Gamma^n p_\delta(s) ds = x_t(\sigma, 0, p_\delta)$  by Theorem 1. This observation leads to the continuity of  $z(t)$  on  $\mathbb{R}$ . Almost the same argument as for  $z(t)$  shows the continuity of the third term of the right-hand side of

(14). The second term of the right-hand side of (14) is identical with  $\Pi^c x_t(0, 0, p_\delta)$  for  $t \in \mathbb{R}$  (cf. (10)), and hence it is continuous on  $\mathbb{R}$ . Thus  $\mathcal{F}_\delta(\psi, y)$  belongs to  $C(\mathbb{R}; X)$ .

By virtue of Theorem 3 combined with (6), we see that

$$\begin{aligned} \|\Pi^c \mathcal{F}_\delta(\psi, y)(t)\|_X e^{-\eta|t|} &\leq e^{-\eta|t|} \left( C e^{\varepsilon|t|} \|\psi\|_X + \left| \int_0^t C C_1 e^{\varepsilon|t-s|} \delta \zeta_*(\delta) ds \right| \right) \\ (15) \quad &\leq C \|\psi\|_X + \frac{C C_1 \delta \zeta_*(\delta)}{\varepsilon} \end{aligned}$$

and that

$$\begin{aligned} \|\Pi^{su} \mathcal{F}_\delta(\psi, y)(t)\|_X e^{-\eta|t|} &\leq e^{-\eta|t|} \left( \int_t^\infty C C_1 e^{\alpha(t-s)} \delta \zeta_*(\delta) ds + \int_{-\infty}^t C C_1 e^{-\alpha(t-s)} \delta \zeta_*(\delta) ds \right) \\ (16) \quad &\leq \left( \frac{2 C C_1 \delta \zeta_*(\delta)}{\alpha} \right) e^{-\eta|t|} \end{aligned}$$

for  $(\psi, y) \in E^c \times Y_\eta$  and  $t \in \mathbb{R}$ . So it follows that

$$\|\mathcal{F}_\delta(\psi, y)(t)\|_X e^{-\eta|t|} \leq C \|\psi\|_X + \delta \zeta_*(\delta) C C_1 \left( \frac{1}{\varepsilon} + \frac{2}{\alpha} \right), \quad t \in \mathbb{R};$$

hence  $\mathcal{F}_\delta(\psi, y)$  belongs to  $Y_\eta$  with  $\|\mathcal{F}_\delta(\psi, y)\|_{Y_\eta} \leq C \|\psi\|_X + \delta \zeta_*(\delta) C C_1 (1/\varepsilon + 2/\alpha)$ . Thus,  $\mathcal{F}_\delta$  defines a map from  $E^c \times Y_\eta$  to  $Y_\eta$ .

(ii) Let  $\psi \in E^c$  and  $y_1, y_2 \in Y_\eta$ . Then by (6), together with (13),

$$\begin{aligned} \|\mathcal{F}_\delta(\psi, y_1) - \mathcal{F}_\delta(\psi, y_2)\|_{Y_\eta} &\leq \sup_{t \in \mathbb{R}} e^{-\eta|t|} \left| \int_0^t C C_1 \zeta_*(\delta) e^{-\varepsilon(t-s)} \|y_1 - y_2\|_{Y_\eta} e^{\eta|s|} ds \right| \\ &\quad + \sup_{t \in \mathbb{R}} e^{-\eta|t|} \int_t^\infty C C_1 \zeta_*(\delta) e^{\alpha(t-s)} \|y_1 - y_2\|_{Y_\eta} e^{\eta|s|} ds \\ &\quad + \sup_{t \in \mathbb{R}} e^{-\eta|t|} \int_{-\infty}^t C C_1 \zeta_*(\delta) e^{-\alpha(t-s)} \|y_1 - y_2\|_{Y_\eta} e^{\eta|s|} ds \\ &\leq \zeta_*(\delta_1) C C_1 \left( \frac{1}{\eta - \varepsilon} + \frac{2}{\alpha + \eta} + \frac{2}{\alpha - \eta} \right) \|y_1 - y_2\|_{Y_\eta} \\ &\leq (1/2) \|y_1 - y_2\|_{Y_\eta}, \end{aligned}$$

so that  $\mathcal{F}_\delta(\psi, \cdot)$  is a contraction map with the required property.  $\square$

In view of Proposition 1 (ii) the map  $\mathcal{F}_\delta(\psi, \cdot)$  has a unique fixed point for each  $\psi \in E^c$ , say  $\Lambda_{*,\delta}(\psi) \in Y^\eta$ , i.e., we have

$$\begin{aligned} \Lambda_{*,\delta}(\psi)(t) &= T^c(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s) \Pi^c \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds \\ (17) \quad &\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s) \Pi^u \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s) \Pi^s \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds \end{aligned}$$

for  $t \in \mathbb{R}$ , whenever  $0 < \delta \leq \delta_1$ .

**Proposition 2.**  $\Lambda_{*,\delta}(\psi)$  satisfies the following:

- (i)  $\|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} \leq 2C\|\psi_1 - \psi_2\|_X$  for  $\psi_1, \psi_2 \in E^c$ .
- (ii)  $\Lambda_{*,\delta}(\psi)(t + \tau) = \Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)))(t)$  holds for  $t, \tau \in \mathbb{R}$ .

*Proof.* (i) By Proposition 1 (ii) it follows that

$$\begin{aligned} \|\Lambda_*(\psi_1) - \Lambda_*(\psi_2)\|_{Y_\eta} &= \|\mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_1)) - \mathcal{F}_\delta(\psi_2, \Lambda_{*,\delta}(\psi_2))\|_{Y_\eta} \\ &\leq \|\mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_1)) - \mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_2))\|_{Y_\eta} \\ &\quad + \|\mathcal{F}_\delta(\psi_1, \Lambda_{*,\delta}(\psi_2)) - \mathcal{F}_\delta(\psi_2, \Lambda_{*,\delta}(\psi_2))\|_{Y_\eta} \\ &\leq (1/2)\|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} + \|T^c(\cdot)(\psi_1 - \psi_2)\|_{Y_\eta}, \end{aligned}$$

so that

$$\begin{aligned} \|\Lambda_*(\psi_1) - \Lambda_*(\psi_2)\|_{Y_\eta} &\leq 2\|T^c(\cdot)(\psi_1 - \psi_2)\|_{Y_\eta} \\ &= 2 \sup_{t \in \mathbb{R}} \|T^c(t)(\psi_1 - \psi_2)\|_X e^{-\eta|t|} \\ &\leq 2 \sup_{t \in \mathbb{R}} (C e^{\varepsilon|t|} \|\psi_1 - \psi_2\|_X) e^{-\eta|t|} \\ &= 2C\|\psi_1 - \psi_2\|_X. \end{aligned}$$

(ii) Given  $\tau \in \mathbb{R}$ , let us set

$$\tilde{\Lambda}(t) := \Lambda_{*,\delta}(\psi)(t + \tau), \quad t \in \mathbb{R}.$$

Obviously,  $\tilde{\Lambda}(\cdot) \in Y_\eta$  and

$$\begin{aligned} \tilde{\Lambda}(t) &= \Lambda_{*,\delta}(\psi)(t + \tau) \\ &= T^c(t + \tau)\psi + \lim_{n \rightarrow \infty} \int_0^{t+\tau} T^c(t + \tau - s) \Pi^c \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds \\ &\quad - \lim_{n \rightarrow \infty} \int_{t+\tau}^\infty T^u(t + \tau - s) \Pi^u \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^{t+\tau} T^s(t + \tau - s) \Pi^s \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds \\ &= T^c(t) \left( T^c(\tau)\psi + \lim_{n \rightarrow \infty} \int_0^\tau T^c(\tau - s) \Pi^c \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds \right) \\ &\quad + \lim_{n \rightarrow \infty} \int_0^t T^c(t - s) \Pi^c \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s + \tau)) ds \\ &\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t - s) \Pi^u \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s + \tau)) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t - s) \Pi^s \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s + \tau)) ds \end{aligned}$$

$$\begin{aligned}
&= T^c(t)(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau))) + \lim_{n \rightarrow \infty} \int_0^t T^c(t-s) \Pi^c \Gamma^n f_\delta(\tilde{\Lambda}(s)) ds \\
&\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s) \Pi^u \Gamma^n f_\delta(\tilde{\Lambda}(s)) ds + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s) \Pi^s \Gamma^n f_\delta(\tilde{\Lambda}(s)) ds \\
&= \mathcal{F}_\delta(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)), \tilde{\Lambda})(t), \quad t \in \mathbb{R},
\end{aligned}$$

that is,  $\tilde{\Lambda}$  is a fixed point of  $\mathcal{F}_\delta(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)), \cdot)$ . The uniqueness of the fixed points yields  $\tilde{\Lambda} = \Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)))$ , and hence

$$\Lambda_{*,\delta}(\psi)(t + \tau) = \tilde{\Lambda}(t) = \Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\psi)(\tau)))(t), \quad t \in \mathbb{R},$$

as desired.  $\square$

For  $\delta \in (0, \delta_1]$  let  $F_{*,\delta} : E^c \rightarrow E^{su}$  be the map defined by  $F_{*,\delta}(\psi) := \Pi^{su} \circ \text{ev}_0 \circ \Lambda_{*,\delta}(\psi)$  for  $\psi \in E^c$ , where  $\text{ev}_0$  is the evaluation map:  $\text{ev}_0(y) := y(0)$  for  $y \in C(\mathbb{R}; X)$ . Since

$$\begin{aligned}
\Lambda_{*,\delta}(\psi)(0) &= \psi - \lim_{n \rightarrow \infty} \int_0^\infty T^u(-s) \Pi^u \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds \\
&\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^0 T^s(-s) \Pi^s \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds,
\end{aligned}$$

it follows that

$$\begin{aligned}
(18) \quad F_{*,\delta}(\psi) &= - \lim_{n \rightarrow \infty} \int_0^\infty T^u(-s) \Pi^u \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds \\
&\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^0 T^s(-s) \Pi^s \Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s)) ds, \quad \psi \in E^c;
\end{aligned}$$

and in particular  $\Lambda_{*,\delta}(\psi)(0) = \psi + F_{*,\delta}(\psi)$  for  $\psi \in E^c$ .

Let us set

$$W_\delta^c := \text{graph } F_{*,\delta} = \{\psi + F_{*,\delta}(\psi) : \psi \in E^c\}.$$

**Proposition 3.** *The map  $F_{*,\delta}$  and its graph  $W_\delta^c$  have the following properties:*

(i)  $F_{*,\delta}$  is (globally) Lipschitz continuous, i.e.,

$$\|F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2)\|_X \leq L(\delta) \|\psi_1 - \psi_2\|_X, \quad \psi_1, \psi_2 \in E^c,$$

where  $L(\delta) := 4C^2C_1\zeta_*(\delta)/(\alpha - \eta)$ .

(ii) Let  $\hat{\phi} \in W_\delta^c$  and  $\tau \in \mathbb{R}$ . Then the solution of  $(E_\delta)$  through  $(\tau, \hat{\phi})$ ,  $x(t; \tau, \hat{\phi}, f_\delta)$ , exists on  $\mathbb{R}$  and

$$x_t(\tau, \hat{\phi}, f_\delta) = \Lambda_{*,\delta}(\hat{\psi})(t - \tau), \quad t \in \mathbb{R},$$

where  $\hat{\psi} = \Pi^c \hat{\phi}$ .



(iii) Moreover for  $\hat{\phi} \in W_\delta^c$  and  $\tau \in \mathbb{R}$ ,

$$\Pi^{su} x_t(\tau, \hat{\phi}, f_\delta) = F_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta)), \quad t \in \mathbb{R}.$$

In particular  $W_\delta^c$  is invariant for  $(E_\delta)$ , that is,  $x_t(\tau, \hat{\phi}, f_\delta) \in W_\delta^c$  for  $t \in \mathbb{R}$ , provided that  $\hat{\phi} \in W_\delta^c$ .

*Proof.* (i) By (18) and Proposition 2 (i)

$$\begin{aligned} \|\Pi^s(F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2))\|_X &\leq \int_{-\infty}^0 CC_1 e^{\alpha s} \zeta_*(\delta) \|\Lambda_{*,\delta}(\psi_1)(s) - \Lambda_{*,\delta}(\psi_2)(s)\|_X ds \\ &\leq \int_{-\infty}^0 CC_1 e^{\alpha s} \zeta_*(\delta) \|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} e^{\eta|s|} ds \\ &= \frac{CC_1 \zeta_*(\delta)}{\alpha - \eta} \|\Lambda_{*,\delta}(\psi_1) - \Lambda_{*,\delta}(\psi_2)\|_{Y_\eta} \\ &\leq \frac{L(\delta)}{2} \|\psi_1 - \psi_2\|_X. \end{aligned}$$

A similar calculation gives

$$\|\Pi^u(F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2))\|_X \leq \frac{L(\delta)}{2} \|\psi_1 - \psi_2\|_X,$$

and therefore

$$\|F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2)\|_X \leq L(\delta) \|\psi_1 - \psi_2\|_X.$$

(ii) Applying Lemma 1 (i), we deduce that  $\Lambda_{*,\delta}(\hat{\psi}) \in C(\mathbb{R}; X_0)$  and that the  $X$ -valued function  $\xi(t) := (\Lambda_{*,\delta}(\hat{\psi})(t))[0]$  ( $t \in \mathbb{R}$ ) satisfies  $\xi_t = \Lambda_{*,\delta}(\hat{\psi})(t)$  for  $t \in \mathbb{R}$  and is a solution of  $(E_\delta)$  on  $\mathbb{R}$  with  $\xi_0 = \Lambda_{*,\delta}(\hat{\psi})(0) = \hat{\psi} + F_{*,\delta}(\hat{\psi}) = \hat{\phi}$ . Let  $x(t) := \xi(t - \tau)$ . Then  $x(t)$  is a solution of  $(E_\delta)$  on  $\mathbb{R}$  with  $x_\tau = \hat{\phi}$ , so that  $x(t) = x(t; \tau, \hat{\phi}, f_\delta)$  for  $t \in \mathbb{R}$ . Consequently,

$$x_t(\tau, \hat{\phi}, f_\delta) = \xi_{t-\tau} = \Lambda_{*,\delta}(\hat{\psi})(t - \tau), \quad t \in \mathbb{R}.$$

(iii) Notice from Proposition 2 (ii) that  $\Lambda_{*,\delta}(\hat{\psi})(t - \tau) = \Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\hat{\psi})(t - \tau)))(0)$  for  $\hat{\psi} := \Pi^c \hat{\phi}$ , which, combined with (ii), yields

$$\begin{aligned} \Pi^{su} x_t(\tau, \hat{\phi}, f_\delta) &= \Pi^{su}(\Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\hat{\psi})(t - \tau)))(0)) \\ &= \Pi^{su}(\Lambda_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta))(0)) \\ &= F_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta)). \end{aligned}$$

The latter part of (iii) is obvious. □

Now assume that  $\Sigma^u = \emptyset$ , i.e.,  $E^u = \{0\}$ . Fix a  $\delta \in (0, \delta_1]$  and let

$$K := CC_1 \zeta_*(\delta), \quad \mu := K + \varepsilon.$$

**Proposition 4.** *Let  $x(t)$  be a solution of  $(E_\delta)$  on an interval  $J := [t_0, t_1]$ . Given  $\tau \in J$ , put  $\hat{\phi} := \Pi^c x_\tau + F_{*,\delta}(\Pi^c x_\tau)$ . Then the following inequalities hold:*

(i) For  $t_0 \leq t \leq \tau$

$$\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq K \int_t^\tau e^{\mu(s-t)} \|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds.$$

(ii) Moreover for  $t_0 \leq t \leq \tau$

$$\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq K \int_t^\tau e^{\mu'(s-t)} \|\xi(s)\|_X ds,$$

where  $\mu' := \mu + KL(\delta)$  and  $\xi(t) := \Pi^s x_t - F_{*,\delta}(\Pi^c x_t)$  for  $t \in \mathbb{R}$ .

For the proof of Proposition 4, we need the following lemma.

**Lemma 2.** Let  $g(t)$ ,  $h(t)$  and  $r(t)$  are real valued continuous functions on the interval  $[t_0, \tau]$  such that  $r(t) \geq 0$  and

$$(19) \quad g(t) \leq \int_t^\tau h(s) ds + \int_t^\tau r(s) g(s) ds$$

for  $t \in [t_0, \tau]$ . Then we have

$$g(t) \leq \int_t^\tau h(s) \exp\left(\int_t^s r(\sigma) d\sigma\right) ds, \quad t \in [t_0, \tau].$$

*Proof.* Put

$$F(t) := \int_t^\tau r(s) g(s) ds, \quad H(t) := \int_t^\tau h(s) ds \quad \text{and} \quad R(t) := \int_t^\tau r(s) ds.$$

By the assumption (19),  $-F'(t) = r(t)g(t) \leq r(t)(F(t) + H(t))$  and hence

$$\frac{d}{dt}(e^{-R(t)} F(t)) = e^{-R(t)} (r(t)F(t) + F'(t)) \geq -r(t)e^{-R(t)} H(t).$$

Since  $F(\tau) = H(\tau) = 0$ ,

$$\begin{aligned} e^{-R(t)} F(t) &\leq \int_t^\tau r(s) e^{-R(s)} H(s) ds \\ &= -e^{-R(t)} H(t) - \int_t^\tau e^{-R(s)} H'(s) ds \\ &= -e^{-R(t)} H(t) + \int_t^\tau e^{-R(s)} h(s) ds, \end{aligned}$$

and hence

$$F(t) + H(t) \leq \int_t^\tau e^{R(t)-R(s)} h(s) ds.$$

So, by using (19) again, we get

$$g(t) \leq F(t) + H(t) \leq \int_t^\tau h(s) \exp\left(\int_t^s r(\sigma) d\sigma\right) ds.$$

This completes the proof. □

*Proof of Proposition 4.* (i) By Proposition 3 (ii) and (iii), the solution  $x(t; \tau, \hat{\phi}, f_\delta)$  exists on  $\mathbb{R}$  and  $\Pi^s x_t(\tau, \hat{\phi}, f_\delta) = F_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta))$  for  $t \in \mathbb{R}$ . Since  $t \leq \tau$ , VCF gives

$$x_\tau(\tau, \hat{\phi}, f_\delta) = T(\tau - t)x_t(\tau, \hat{\phi}, f_\delta) + \lim_{n \rightarrow \infty} \int_t^\tau T(\tau - s)\Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds,$$

in particular

$$\Pi^c x_\tau(\tau, \hat{\phi}, f_\delta) = T^c(\tau - t)\Pi^c x_t(\tau, \hat{\phi}, f_\delta) + \lim_{n \rightarrow \infty} \int_t^\tau T^c(\tau - s)\Pi^c \Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds.$$

By the group property of  $\{T^c(t)\}_{t \in \mathbb{R}}$

$$(20) \quad \begin{aligned} \Pi^c x_t(\tau, \hat{\phi}, f_\delta) &= T^c(t - \tau)\Pi^c x_\tau(\tau, \hat{\phi}, f_\delta) \\ &\quad - \lim_{n \rightarrow \infty} \int_t^\tau T^c(t - s)\Pi^c \Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds. \end{aligned}$$

Similarly for the solution  $x(t)$

$$\Pi^c x_t = T^c(t - \tau)\Pi^c x_\tau - \lim_{n \rightarrow \infty} \int_t^\tau T^c(t - s)\Pi^c \Gamma^n f_\delta(x_s)ds.$$

Noting that  $\Pi^c x_\tau(\tau, \hat{\phi}, f_\delta) = \Pi^c \hat{\phi} = \Pi^c x_\tau$ , we obtain

$$\begin{aligned} \|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X &\leq \int_t^\tau CC_1 e^{\varepsilon|t-s|} \zeta_*(\delta) \|x_s - x_s(\tau, \hat{\phi}, f_\delta)\|_X ds \\ &\leq \int_t^\tau K e^{\varepsilon(s-t)} (\|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X \\ &\quad + \|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X) ds, \end{aligned}$$

so that

$$\begin{aligned} e^{\varepsilon t} \|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X &\leq \int_t^\tau K e^{\varepsilon s} \|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds \\ &\quad + \int_t^\tau K e^{\varepsilon s} \|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X ds \end{aligned}$$

for  $t_0 \leq t \leq \tau$ . Applying Lemma 2, we get

$$e^{\varepsilon t} \|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq \int_t^\tau K e^{K(s-t)} e^{\varepsilon s} \|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds,$$

which implies (i).

(ii) By virtue of Proposition 3 (iii) and (i)

$$\begin{aligned} \|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X &\leq \|\Pi^s x_s - F_{*,\delta}(\Pi^c x_s)\|_X \\ &\quad + \|F_{*,\delta}(\Pi^c x_s) - F_{*,\delta}(\Pi^c x_s(\tau, \hat{\phi}, f_\delta))\|_X \\ &\leq \|\xi(s)\|_X + L(\delta) \|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X \end{aligned}$$

for  $s \in J$ . Hence it follows from (i) that

$$\begin{aligned} e^{\mu t} \|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X &\leq K \int_t^\tau e^{\mu s} \|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds \\ &\leq \int_t^\tau K e^{\mu s} \|\xi(s)\|_X ds \\ &\quad + \int_t^\tau K L(\delta) e^{\mu s} \|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X ds. \end{aligned}$$

Then another application of Lemma 2 readily yields (ii).  $\square$

Recall that

$$(21) \quad K := CC_1 \zeta_*(\delta), \quad \mu := K + \varepsilon, \quad \mu' := \mu + KL(\delta) = K(1 + L(\delta)) + \varepsilon.$$

**Proposition 5.** *Assume that  $\Sigma^u = \emptyset$  and  $x(t)$  is a solution of  $(E_\delta)$  on  $J = [t_0, t_1]$ . Define  $\hat{x}_t \in W_\delta^c$  by  $\hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t)$  for  $t \in J$ , and set  $y(s; t) := \Pi^c x_s(t, \hat{x}_t, f_\delta)$  for  $t \in J$  and  $s \leq t$ . Then the following inequality holds:*

$$\|y(s; t) - y(s; t_0)\|_X \leq K \int_{t_0}^t e^{\mu'(\theta-s)} \|\xi(\theta)\|_X d\theta, \quad s \leq t_0,$$

where  $\xi(\theta) := \Pi^s x_\theta - F_{*,\delta}(\Pi^c x_\theta)$  for  $\theta \in [t_0, t]$ .

*Proof.* Suppose that  $s \leq t_0$ . By the same reasoning as (20)

$$(22) \quad \Pi^c x_s(t, \hat{x}_t, f_\delta) = T^c(s-t) \Pi^c \hat{x}_t - \lim_{n \rightarrow \infty} \int_s^t T^c(s-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma(t, \hat{x}_t, f_\delta)) d\sigma.$$

Applying VCF to  $x_t$  and using  $\Pi^c \hat{x}_\tau = \Pi^c x_\tau$  ( $\tau \in J$ ), we deduce that

$$\Pi^c \hat{x}_t = T^c(t-t_0) \Pi^c \hat{x}_{t_0} + \lim_{n \rightarrow \infty} \int_{t_0}^t T^c(t-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma) d\sigma,$$

and, thus, (22) becomes

$$\begin{aligned} \Pi^c x_s(t, \hat{x}_t, f_\delta) &= T^c(s-t_0) \Pi^c \hat{x}_{t_0} + \lim_{n \rightarrow \infty} \int_{t_0}^t T^c(s-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma) d\sigma \\ &\quad - \lim_{n \rightarrow \infty} \int_s^t T^c(s-\sigma) \Pi^c \Gamma^n f_\delta(x_\sigma(t, \hat{x}_t, f_\delta)) d\sigma, \quad t \in J. \end{aligned}$$

Therefore

$$\begin{aligned}
 \|y(s; t) - y(s; t_0)\|_X &= \|\Pi^c x_s(t, \hat{x}_t, f_\delta) - \Pi^c x_s(t_0, \hat{x}_{t_0}, f_\delta)\|_X \\
 &= \left\| \lim_{n \rightarrow \infty} \int_{t_0}^t T^c(s - \sigma) \Pi^c \Gamma^n f_\delta(x_\sigma) d\sigma \right. \\
 &\quad \left. - \lim_{n \rightarrow \infty} \int_s^t T^c(s - \sigma) \Pi^c \Gamma^n f_\delta(x_\sigma(t, \hat{x}_t, f_\delta)) d\sigma \right. \\
 &\quad \left. + \lim_{n \rightarrow \infty} \int_s^{t_0} T^c(s - \sigma) \Pi^c \Gamma^n f_\delta(x_\sigma(t_0, \hat{x}_{t_0}, f_\delta)) d\sigma \right\|_X \\
 &\leq \int_{t_0}^t C C_1 e^{\varepsilon|s-\sigma|} \zeta_*(\delta) \|x_\sigma - x_\sigma(t, \hat{x}_t, f_\delta)\|_X d\sigma \\
 &\quad + \int_s^{t_0} C C_1 e^{\varepsilon|s-\sigma|} \zeta_*(\delta) \|x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - x_\sigma(t, \hat{x}_t, f_\delta)\|_X d\sigma.
 \end{aligned} \tag{23}$$

Observe that

$$\begin{aligned}
 \|x_\sigma - x_\sigma(t, \hat{x}_t, f_\delta)\|_X &\leq \|\Pi^s x_\sigma - \Pi^s x_\sigma(t, \hat{x}_t, f_\delta)\|_X + \|\Pi^c x_\sigma - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X \\
 &\leq \|\Pi^s x_\sigma - F_{*,\delta}(\Pi^c x_\sigma)\|_X + \|F_{*,\delta}(\Pi^c x_\sigma) - F_{*,\delta}(\Pi^c x_\sigma(t, \hat{x}_t, f_\delta))\|_X \\
 &\quad + \|\Pi^c x_\sigma - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X \\
 &\leq \|\xi(\sigma)\|_X + (1 + L(\delta)) \|\Pi^c x_\sigma - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X,
 \end{aligned} \tag{24}$$

where we used Proposition 3 (i) and (iii). Note also that

$$\begin{aligned}
 \|x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - x_\sigma(t, \hat{x}_t, f_\delta)\|_X &\leq \|\Pi^s x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - \Pi^s x_\sigma(t, \hat{x}_t, f_\delta)\|_X \\
 &\quad + \|\Pi^c x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X \\
 &= \|F_{*,\delta}(\Pi^c x_\sigma(t_0, \hat{x}_{t_0}, f_\delta)) - F_{*,\delta}(\Pi^c x_\sigma(t, \hat{x}_t, f_\delta))\|_X \\
 &\quad + \|\Pi^c x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X \\
 &\leq (1 + L(\delta)) \|\Pi^c x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X \\
 &= (1 + L(\delta)) \|y(\sigma; t) - y(\sigma; t_0)\|_X.
 \end{aligned} \tag{25}$$

In view of (23), (24) and (25), combined with Proposition 4 (ii), we deduce

$$\begin{aligned}
 \|y(s; t) - y(s; t_0)\|_X &\leq \int_{t_0}^t K e^{\varepsilon(\sigma-s)} (\|\xi(\sigma)\|_X + (1 + L(\delta)) \|\Pi^c x_\sigma - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X) d\sigma \\
 &\quad + \int_s^{t_0} K e^{\varepsilon(\sigma-s)} (1 + L(\delta)) \|y(\sigma; t) - y(\sigma; t_0)\|_X d\sigma \\
 &\leq \int_{t_0}^t K e^{\varepsilon(\sigma-s)} \|\xi(\sigma)\|_X d\sigma \\
 &\quad + \int_{t_0}^t K e^{\varepsilon(\sigma-s)} (1 + L(\delta)) \left( K \int_\sigma^t e^{\mu'(\tau-\sigma)} \|\xi(\tau)\|_X d\tau \right) d\sigma
 \end{aligned}$$

$$(26) \quad + \int_s^{t_0} K e^{\varepsilon(\sigma-s)} (1 + L(\delta)) \|y(\sigma; t) - y(\sigma; t_0)\|_X d\sigma.$$

Notice that the second term of the right-hand side becomes

$$K \int_{t_0}^t (e^{\varepsilon(t_0-s)+\mu'(\sigma-t_0)} - e^{\varepsilon(\sigma-s)}) \|\xi(\sigma)\|_X d\sigma$$

because of (21). So we see from (26) that for  $s \leq t_0$

$$\begin{aligned} e^{\varepsilon s} \|y(s; t) - y(s; t_0)\|_X &\leq K \int_{t_0}^t e^{(\varepsilon-\mu')t_0+\mu'\sigma} \|\xi(\sigma)\|_X d\sigma \\ &\quad + K(1 + L(\delta)) \int_s^{t_0} e^{\varepsilon\sigma} \|y(\sigma; t) - y(\sigma; t_0)\|_X d\sigma. \end{aligned}$$

By Gronwall's inequality and (21)

$$\begin{aligned} e^{\varepsilon s} \|y(s; t) - y(s; t_0)\|_X &\leq \left( K \int_{t_0}^t e^{(\varepsilon-\mu')t_0+\mu'\sigma} \|\xi(\sigma)\|_X d\sigma \right) e^{K(1+L(\delta))(t_0-s)} \\ &= K e^{-(\mu'-\varepsilon)s} \int_{t_0}^t e^{\mu'\sigma} \|\xi(\sigma)\|_X d\sigma, \end{aligned}$$

and therefore

$$\|y(s; t) - y(s; t_0)\|_X \leq K \int_{t_0}^t e^{\mu'(\sigma-s)} \|\xi(\sigma)\|_X d\sigma, \quad s \leq t_0,$$

as required.  $\square$

**Proposition 6.** Assume that  $\Sigma^u = \emptyset$  and  $\delta \in (0, \delta_1]$  satisfies

$$(27) \quad \max \left( \mu', \frac{K(\alpha - \varepsilon)}{\alpha - \mu'} \right) < \alpha.$$

If  $x(t)$  is a solution of  $(E_\delta)$  on  $J = [t_0, t_1]$ , then the function  $\xi(t) := \Pi^s x_t - F_{*,\delta}(\Pi^c x_t)$  satisfies the inequality

$$\|\xi(t)\|_X \leq C \|\xi(t_0)\|_X e^{-\beta_0(t-t_0)}, \quad t \in J,$$

where  $\beta_0 := \alpha - K(\alpha - \varepsilon)/(\alpha - \mu') > 0$ . If in particular  $J = [t_0, \infty)$ ,  $\text{dist}(x_t, W_\delta^\varepsilon)$  tends to 0 exponentially as  $t \rightarrow \infty$ .

*Proof.* Observe that for  $t \in J$

$$\begin{aligned} \xi(t) - T^s(t - t_0)\xi(t_0) &= \Pi^s x_t - F_{*,\delta}(\Pi^c x_t) - T^s(t - t_0)(\Pi^s x_{t_0} - F_{*,\delta}(\Pi^c x_{t_0})) \\ &= \Pi^s(x_t - T(t - t_0)x_{t_0}) - F_{*,\delta}(\Pi^c x_t) + T^s(t - t_0)F_{*,\delta}(\Pi^c x_{t_0}) \\ &= \lim_{n \rightarrow \infty} \int_{t_0}^t T^s(t - s) \Pi^s \Gamma^n f_\delta(x_s) ds \quad (\text{by VCF}) \\ &\quad - \lim_{n \rightarrow \infty} \int_{-\infty}^0 T^s(-s) \Pi^s \Gamma^n f_\delta(\Lambda_{*,\delta}(\Pi^c x_t)(s)) ds \end{aligned}$$

$$\begin{aligned}
 & + \lim_{n \rightarrow \infty} \int_{-\infty}^0 T^s(t - t_0 - s) \Pi^s \Gamma^n f_\delta(\Lambda_{*,\delta}(\Pi^c x_{t_0})(s)) ds \\
 & = \lim_{n \rightarrow \infty} \int_{t_0-t}^0 T^s(-s) \Pi^s \Gamma^n f_\delta(x_{s+t}) ds \\
 & \quad - \lim_{n \rightarrow \infty} \int_{-\infty}^0 T^s(-s) \Pi^s \Gamma^n f_\delta(\Lambda_{*,\delta}(\Pi^c x_t)(s)) ds \\
 & \quad + \lim_{n \rightarrow \infty} \int_{-\infty}^{t_0-t} T^s(-s) \Pi^s \Gamma^n f_\delta(\Lambda_{*,\delta}(\Pi^c x_{t_0})(t - t_0 + s)) ds \\
 & = \lim_{n \rightarrow \infty} \int_{t_0-t}^0 T^s(-s) \Pi^s \Gamma^n (f_\delta(x_{s+t}) - f_\delta(\Lambda_{*,\delta}(\Pi^c x_t)(s))) ds \\
 & \quad + \lim_{n \rightarrow \infty} \int_{-\infty}^{t_0-t} T^s(-s) \Pi^s \Gamma^n (f_\delta(\Lambda_{*,\delta}(\Pi^c x_{t_0})(t - t_0 + s)) \\
 & \quad - f_\delta(\Lambda_{*,\delta}(\Pi^c x_t)(s))) ds.
 \end{aligned}$$

If we set  $\hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t)$  for  $t \in J$ , by Proposition 3 (ii)  $\Lambda_{*,\delta}(\Pi^c x_t)(s) = x_s(0, \hat{x}_t, f_\delta) = x_{s+t}(t, \hat{x}_t, f_\delta)$  and  $\Lambda_{*,\delta}(\Pi^c x_{t_0})(t - t_0 + s) = x_{t-t_0+s}(0, \hat{x}_{t_0}, f_\delta) = x_{s+t}(t_0, \hat{x}_{t_0}, f_\delta)$  in particular for  $s \in \mathbb{R}^-$ . So

$$\begin{aligned}
 \xi(t) & = T^s(t - t_0) \xi(t_0) + \lim_{n \rightarrow \infty} \int_{t_0-t}^0 T^s(-s) \Pi^s \Gamma^n (f_\delta(x_{s+t}) - f_\delta(x_{s+t}(t, \hat{x}_t, f_\delta))) ds \\
 & \quad + \lim_{n \rightarrow \infty} \int_{-\infty}^{t_0-t} T^s(-s) \Pi^s \Gamma^n (f_\delta(x_{s+t}(t_0, \hat{x}_{t_0}, f_\delta)) - f_\delta(x_{s+t}(t, \hat{x}_t, f_\delta))) ds,
 \end{aligned}$$

and thus

$$\begin{aligned}
 \|\xi(t)\|_X & \leq C e^{-\alpha(t-t_0)} \|\xi(t_0)\|_X + \int_{t_0-t}^0 K e^{\alpha s} \|x_{s+t} - x_{s+t}(t, \hat{x}_t, f_\delta)\|_X ds \\
 & \quad + \int_{-\infty}^{t_0-t} K e^{\alpha s} \|x_{s+t}(t_0, \hat{x}_{t_0}, f_\delta) - x_{s+t}(t, \hat{x}_t, f_\delta)\|_X ds \\
 & = C e^{-\alpha(t-t_0)} \|\xi(t_0)\|_X + \int_{t_0}^t K e^{\alpha(\theta-t)} \|x_\theta - x_\theta(t, \hat{x}_t, f_\delta)\|_X d\theta \\
 & \quad + \int_{-\infty}^{t_0} K e^{\alpha(\theta-t)} \|x_\theta(t_0, \hat{x}_{t_0}, f_\delta) - x_\theta(t, \hat{x}_t, f_\delta)\|_X d\theta.
 \end{aligned}$$

Since  $x_\theta(t, \hat{x}_t, f_\delta)$  ( $t \in J$ ) can be written as

$$\begin{aligned}
 x_\theta(t, \hat{x}_t, f_\delta) & = \Pi^c x_\theta(t, \hat{x}_t, f_\delta) + \Pi^s x_\theta(t, \hat{x}_t, f_\delta) \\
 & = \Pi^c x_\theta(t, \hat{x}_t, f_\delta) + F_{*,\delta}(\Pi^c x_\theta(t, \hat{x}_t, f_\delta)), \quad \theta \in \mathbb{R}
 \end{aligned}$$

(Proposition 3 (iii)), it follows from Proposition 3 (i) and Proposition 5 that for  $\theta \leq t_0$

$$\begin{aligned}
\|x_\theta(t_0, \hat{x}_{t_0}, f_\delta) - x_\theta(t, \hat{x}_t, f_\delta)\|_X &\leq \|\Pi^c x_\theta(t_0, \hat{x}_{t_0}, f_\delta) - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\
&\quad + \|F_{*,\delta}(\Pi^c x_\theta(t_0, \hat{x}_{t_0}, f_\delta)) - F_{*,\delta}(\Pi^c x_\theta(t, \hat{x}_t, f_\delta))\|_X \\
&\leq (1 + L(\delta)) \|\Pi^c x_\theta(t_0, \hat{x}_{t_0}, f_\delta) - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\
&= (1 + L(\delta)) \|y(\theta; t) - y(\theta; t_0)\|_X \\
&\leq (1 + L(\delta)) K \int_{t_0}^t e^{\mu'(\tau-\theta)} \|\xi(\tau)\|_X d\tau,
\end{aligned}$$

where  $y(\theta; t)$  ( $t \in J$ ) is the one in Proposition 5. On the other hand, for  $t_0 \leq \theta \leq t$

$$\begin{aligned}
\|x_\theta - x_\theta(t, \hat{x}_t, f_\delta)\|_X &\leq \|\Pi^s x_\theta - \Pi^s x_\theta(t, \hat{x}_t, f_\delta)\|_X + \|\Pi^c x_\theta - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\
&\leq \|\Pi^s x_\theta - F_{*,\delta}(\Pi^c x_\theta)\|_X + \|F_{*,\delta}(\Pi^c x_\theta) - F_{*,\delta}(\Pi^c x_\theta(t, \hat{x}_t, f_\delta))\|_X \\
&\quad + \|\Pi^c x_\theta - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\
&\leq \|\xi(\theta)\|_X + (1 + L(\delta)) \|\Pi^c x_\theta - \Pi^c x_\theta(t, \hat{x}_t, f_\delta)\|_X \\
&\leq \|\xi(\theta)\|_X + (1 + L(\delta)) K \int_{\theta}^t e^{\mu'(\sigma-\theta)} \|\xi(\sigma)\|_X d\sigma,
\end{aligned}$$

where we used Proposition 3 (i), (iii) and Proposition 4 (ii). Thus we have

$$\begin{aligned}
\|\xi(t)\|_X &\leq C e^{-\alpha(t-t_0)} \|\xi(t_0)\|_X \\
&\quad + \int_{t_0}^t K e^{\alpha(\theta-t)} \left( \|\xi(\theta)\|_X + (1 + L(\delta)) K \int_{\theta}^t e^{\mu'(\sigma-\theta)} \|\xi(\sigma)\|_X d\sigma \right) d\theta \\
&\quad + \int_{-\infty}^{t_0} K e^{\alpha(\theta-t)} (1 + L(\delta)) K \left( \int_{t_0}^t e^{\mu'(\tau-\theta)} \|\xi(\tau)\|_X d\tau \right) d\theta \\
&= C e^{-\alpha(t-t_0)} \|\xi(t_0)\|_X + \left( K + \frac{K^2(1 + L(\delta))}{\alpha - \mu'} \right) \int_{t_0}^t e^{\alpha(\sigma-t)} \|\xi(\sigma)\|_X d\sigma,
\end{aligned}$$

so that

$$e^{\alpha t} \|\xi(t)\|_X \leq C e^{\alpha t_0} \|\xi(t_0)\|_X + \hat{K} \int_{t_0}^t e^{\alpha \sigma} \|\xi(\sigma)\|_X d\sigma,$$

where  $\hat{K} := K + K^2(1 + L(\delta))/(\alpha - \mu')$ . An application of Gronwall's inequality gives

$$e^{\alpha t} \|\xi(t)\|_X \leq C e^{\alpha t_0} \|\xi(t_0)\|_X e^{\hat{K}(t-t_0)},$$

and hence

$$\|\xi(t)\|_X \leq C \|\xi(t_0)\|_X e^{-(\alpha - \hat{K})(t-t_0)}, \quad t \in J,$$

which is the desired one because in view of (21)

$$\hat{K} = K \frac{\alpha - \varepsilon}{\alpha - \mu'} = \alpha - \beta_0.$$

The latter part of the proposition is evident. This completes the proof.  $\square$



*Proof of Theorem 4.* The properties (ii) and (iii) of Theorem 4 are now immediate consequences of Propositions 3 and 6, respectively. We verify the property (i). Observe that  $Y_\eta$  is a subspace of  $Y_{\eta'}$  if  $\eta < \eta' < \alpha$ , and denote the inclusion map by  $\mathcal{J} : Y_\eta \rightarrow Y_{\eta'}$ . It will be shown, in the appendix (Proposition 10), that  $\mathcal{J}\Lambda_{*,\delta}$  is  $C^1$  smooth as a map from  $E^c$  to  $Y_{\eta'}$ ; and hence  $F_{*,\delta} = \Pi^{su} \circ \text{ev}_0 \circ \mathcal{J}\Lambda_{*,\delta}$  is also  $C^1$  smooth. Moreover, the relation

$$[D(\mathcal{J}\Lambda_{*,\delta})(0)](t)\psi = T^c(t)\psi, \quad \psi \in E^c, \quad t \in \mathbb{R}$$

holds since  $Df_\delta(0) = Df(0) = 0$  (cf. (41) and (42)). In particular

$$DF_{*,\delta}(0)\psi = D(\Pi^{su} \circ \text{ev}_0 \circ \mathcal{J}\Lambda_{*,\delta})(0)\psi = \Pi^{su}T^c(0)\psi = \Pi^{su}\psi = 0, \quad \psi \in E^c,$$

so that  $DF_{*,\delta}(0) = 0$ , which implies (i).  $\square$

**3.2. Stability for integral equations via the central equation.** In this subsection, introducing some ordinary differential equation which we call the central equation, we will study stability properties for the zero solution of Eq. (E).

Assume that  $\Sigma^c \neq \emptyset$ . Let  $\{\phi_1, \dots, \phi_{d_c}\}$  be a basis for  $E^c$ , where  $d_c$  is the dimension of  $E^c$ . Then based on the formal adjoint theory for Eq. (5) developed in [24] (also refer to [23]) where the formal adjoint theory is accomplished for Volterra difference equations), one can consider its dual basis as elements in the Banach space

$$X^\sharp := L^1_\rho(\mathbb{R}^+; (\mathbb{C}^*)^m) = \{\psi : \mathbb{R}^+ \rightarrow (\mathbb{C}^*)^m : \psi(\tau)e^{-\rho\tau} \text{ is integrable on } \mathbb{R}^+\}$$

with norm

$$\|\psi\|_{X^\sharp} := \int_0^\infty |\psi(\tau)|e^{-\rho\tau} d\tau, \quad \psi \in X^\sharp,$$

where  $(\mathbb{C}^*)^m$  is the space of  $m$ -dimensional row vectors with complex components equipped with the norm which is compatible with the one in  $\mathbb{C}^m$ , that is,  $|z^*z| \leq |z^*||z|$  for  $z^* \in (\mathbb{C}^*)^m$  and  $z \in \mathbb{C}^m$ . To be more precise, if we set

$$\langle\langle \psi, \phi \rangle\rangle := \int_{-\infty}^0 \left( \int_\theta^0 \psi(\xi - \theta)K(-\theta)\phi(\xi)d\xi \right) d\theta, \quad (\psi, \phi) \in X^\sharp \times X,$$

then this pairing defines a bounded bilinear form on  $X^\sharp \times X$  with the property

$$|\langle\langle \psi, \phi \rangle\rangle| \leq \|K\|_{\infty, \rho} \|\psi\|_{X^\sharp} \|\phi\|_X, \quad (\psi, \phi) \in X^\sharp \times X;$$

here we recall that  $\|K\|_{\infty, \rho} = \text{ess sup}\{\|K(t)\|e^{\rho t} : t \geq 0\}$ . Then there exist  $\{\psi_1, \dots, \psi_{d_c}\}$ , elements of  $X^\sharp$ , such that  $\langle\langle \psi_i, \phi_j \rangle\rangle = 1$  if  $i = j$  and 0 otherwise, and  $\langle\langle \psi_i, \phi \rangle\rangle = 0$  for  $\phi \in E^s$  and  $i = 1, 2, \dots, d_c$ ; we call  $\{\psi_1, \dots, \psi_{d_c}\}$  the dual basis of  $\{\phi_1, \dots, \phi_{d_c}\}$ . (See [24] for details.)

Denote by  $\Phi_c$  and  $\Psi_c$ ,  $(\phi_1, \dots, \phi_{d_c})$  and  ${}^t(\psi_1, \dots, \psi_{d_c})$ , the transpose of  $(\psi_1, \dots, \psi_{d_c})$ , respectively. Then, for any  $\phi \in X$  the coordinate of its  $E^c$ -component with respect to the basis  $\{\phi_1, \dots, \phi_{d_c}\}$ , or  $\Phi_c$  for short, is given by

$$\langle\langle \Psi_c, \phi \rangle\rangle := {}^t(\langle\langle \psi_1, \phi \rangle\rangle, \dots, \langle\langle \psi_{d_c}, \phi \rangle\rangle) \in \mathbb{C}^{d_c},$$

and therefore the projection  $\Pi^c$  is expressed, in terms of the basis  $\Phi_c$  and its dual basis  $\Psi_c$ , by

$$(28) \quad \Pi^c \phi = \Phi_c \langle \Psi_c, \phi \rangle, \quad \phi \in X.$$

Since  $\{T^c(t)\}_{t \geq 0}$  is a strongly continuous semigroup on the finite dimensional space  $E^c$ , there exists a  $d_c \times d_c$  matrix  $G_c$  such that

$$(29) \quad T^c(t)\Phi_c = \Phi_c e^{tG_c}, \quad t \geq 0,$$

and  $\sigma(G_c)$ , the spectrum of  $G_c$ , is identical with  $\Sigma^c$ . The  $E^c$ -components of solutions of Eq.  $(E_\delta)$  can be described by a certain ordinary differential equation in  $\mathbb{C}^{d_c}$ . More precisely, let  $x(t)$  be a solution of Eq.  $(E_\delta)$  through  $(\sigma, \phi)$ , that is,  $x(t) = x(t; \sigma, \phi, f)$ . If we denote by  $z_c(t)$  the component of  $\Pi^c x_t$  with respect to the basis  $\Phi_c$ , that is,  $\Phi_c z_c(t) := \Pi^c x_t$ , or  $z_c(t) := \langle \Psi_c, x_t \rangle$ , then by virtue of [22, Theorem 7]  $z_c(t)$  satisfies the ordinary differential equation

$$(30) \quad \dot{z}_c(t) = G_c z_c(t) + H_c f_\delta(\Phi_c z_c(t) + \Pi^{su} x_t),$$

where  $H_c$  is the  $d_c \times m$  matrix such that

$$H_c x := \lim_{n \rightarrow \infty} \langle \Psi_c, \Gamma^n x \rangle, \quad x \in \mathbb{C}^m.$$

Thus the  $E^c$ -components of solutions of Eq.  $(E_\delta)$  are determined by solutions of (30).

In connection with Eq. (30), let us consider the ordinary differential equations on  $\mathbb{C}^{d_c}$

$$(CE_\delta) \quad \dot{z}(t) = G_c z(t) + H_c f_\delta(\Phi_c z(t) + F_{*,\delta}(\Phi_c z(t)))$$

and

$$(CE) \quad \dot{z}(t) = G_c z(t) + H_c f(\Phi_c z(t) + F_*(\Phi_c z(t))).$$

We call Eq.  $(CE)$  (resp. Eq.  $(CE_\delta)$ ) the central equation of  $(E)$  (resp.  $(E_\delta)$ ).

**Proposition 7.** *The following statements hold true:*

- (i) *Let  $x$  be a solution of Eq.  $(E_\delta)$  on an interval  $J$  such that  $x_t \in W_\delta^c$  ( $t \in J$ ). Then the function  $z_c(t) := \langle \Psi_c, x_t \rangle$  satisfies the equation  $(CE_\delta)$  on  $J$ . Conversely, if  $z(t)$  satisfies the equation  $(CE_\delta)$  on an interval  $J$ , then there exists a unique solution  $x$  of Eq.  $(E_\delta)$  on  $J$  such that  $x_t \in W_\delta^c$  and  $\Pi^c x_t = \Phi_c z(t)$  on  $J$ .*
- (ii) *Let  $x$  be a solution of Eq.  $(E)$  on an interval  $J$  such that  $x_t \in W_{\text{loc}}^c(r, \delta)$  ( $t \in J$ ). Then the function  $z_c(t) := \langle \Psi_c, x_t \rangle$  satisfies the equation  $(CE)$  on  $J$ , together with the inequality  $\sup_{t \in J} \|\Phi_c z_c(t)\|_X \leq r$ . Conversely, if  $z(t)$  satisfies the equation  $(CE)$  on an interval  $J$  together with the inequality  $\sup_{t \in J} \|\Phi_c z(t)\|_X \leq r$ , then there exists a unique solution  $x$  of Eq.  $(E)$  on  $J$  such that  $x_t \in W_{\text{loc}}^c(r, \delta)$  and  $\Pi^c x_t = \Phi_c z(t)$  on  $J$ .*

*Proof.* Let us recall that whenever  $\phi \in X$  satisfies  $\|\Pi^c \phi\|_X \leq r$ ,  $\phi \in W_\delta^c$  if and only if  $\phi \in W_{\text{loc}}^c(r, \delta)$ ; and consequently  $F_{*,\delta}(\Pi^c \phi) = F_*(\Pi^c \phi)$  and  $f_\delta(\Pi^c \phi + F_{*,\delta}(\Pi^c \phi)) = f(\Pi^c \phi + F_*(\Pi^c \phi))$ . Therefore  $z(t)$  satisfying  $\sup_{t \in J} \|\Phi_c z(t)\|_X \leq r$  is a solution of  $(CE_\delta)$  on  $J$  if and only if it is a solution of  $(CE)$  on  $J$ . Thus (ii) is a direct consequence of (i); so, in what follows, we will prove (i) only.

The former part of (i) directly follows from Proposition 3 (iii). Conversely, let  $z(t)$  be a solution of  $(CE_\delta)$  on  $J$ . Pick  $\tau \in J$  and set  $\hat{\phi} = \Phi_c z(\tau) + F_{*,\delta}(\Phi_c z(\tau))$ . Then it follows from Proposition 3 (iii) again that  $x(t) := x(t; \tau, \hat{\phi}, f_\delta)$  is a solution of  $(E_\delta)$  on  $J$  such that  $x_t \in W_\delta^c$  for  $t \in J$ . By the former part, the function  $z_c(t)$ , defined by  $\Phi_c z_c(t) = \Pi^c x_t$ , is also a solution of  $(CE_\delta)$  satisfying  $z_c(\tau) = \langle \Psi_c, \hat{\phi} \rangle = \langle \Psi_c, \Phi_c z(\tau) \rangle = z(\tau)$ . The uniqueness of solutions of  $(CE_\delta)$  yields  $z_c(t) = z(t)$  for  $t \in J$  and hence  $\Pi^c x_t = \Phi_c z_c(t) = \Phi_c z(t)$  for  $t \in J$ .  $\square$

Since  $f(0) = f_\delta(0) = 0$ , both equations  $(CE)$  and  $(CE_\delta)$  (as well as  $(E)$  and  $(E_\delta)$ ) possess the zero solution. Notice that the zero solution of  $(CE)$  (resp.  $(E)$ ) is uniformly asymptotically stable if and only if the zero solution of  $(CE_\delta)$  (resp.  $(E_\delta)$ ) is uniformly asymptotically stable. Likewise, the zero solution of  $(CE)$  (resp.  $(E)$ ) is unstable if and only if the zero solution of  $(CE_\delta)$  (resp.  $(E_\delta)$ ) is unstable. Here, for the definition of several stability properties utilized in this paper, we refer readers to the books [33, 12].

Now suppose that  $\Sigma^u = \emptyset$ . Then the dynamics near the zero solution of  $(E)$  is determined by the dynamics near  $z_c = 0$  of  $(CE)$  in the following sense.

**Theorem 6.** *Assume that  $\Sigma^u = \emptyset$ . If the zero solution of  $(CE)$  is uniformly asymptotically stable (resp. unstable), then the zero solution of  $(E)$  is also uniformly asymptotically stable (resp. unstable).*

*Proof.* By the fact stated in the preceding paragraph of the theorem, it is sufficient to establish that the uniform asymptotic stability (resp. instability) of the zero solution of  $(CE_\delta)$  implies the uniform asymptotic stability (resp. instability) of the zero solution of  $(E_\delta)$ .

If the zero solution of  $(CE_\delta)$  is unstable, the instability of the zero solution of  $(E_\delta)$  immediately follows from the invariance of  $W_\delta^c$  (Proposition 3 (iii)). In what follows, under the assumption that the zero solution of  $(CE_\delta)$  is uniformly asymptotically stable, we will establish the uniform asymptotic stability of the zero solution of  $(E_\delta)$ , employing an idea utilized for the stability problems of parabolic partial differential equations in [12, Theorem 6.1.4].

By virtue of [12, Theorem 4.2.1], there exist positive constants  $a$ ,  $\bar{K}$  and a Liapunov function  $V$  defined on  $S_a := \{y \in \mathbb{C}^{d_c} : |y| \leq a\}$  satisfying the following properties:

(i) There exists a  $b \in C(\mathbb{R}^+; \mathbb{R}^+)$  which is strictly increasing with  $b(0) = 0$  and

$$b(|y|) \leq V(y) \leq |y| \quad \text{for } y \in S_a.$$

(ii)  $|V(y) - V(z)| \leq \bar{K}|y - z|$  for  $y, z \in S_a$ .

(iii)  $\dot{V}(z) \leq -V(z)$  for  $z \in S_a$ , where  $\dot{V}(z)$  is defined by

$$\dot{V}(z) := \limsup_{h \rightarrow +0} \frac{V(y(h)) - V(z)}{h},$$

$y(h)$  being the solution of  $(CE_\delta)$  with  $y(0) = z$ .

Choose a positive number  $\tau_0$  such that

$$(31) \quad e^{-\tau_0} \leq \frac{1}{2} \quad \text{and} \quad Ce^{-\beta_0 \tau_0} \leq \frac{1}{4},$$

where  $\beta_0$  is the one in Proposition 6, and we may assume that  $\beta_0 > \mu'$ , taking  $\delta$  so small if necessary. Put  $K_\infty := \|K\|_{\infty, \rho}$  and take a positive number  $P$  in such a way that

$$(32) \quad P > \max \left( 1, \frac{4}{\beta_0 - \mu'} \bar{K} K K_\infty \|\Psi_c\| \right),$$

and set

$$a_0 := \frac{ae^{-\eta \tau_0}}{4CK_\infty \|\Psi_c\|},$$

where  $\|\Psi_c\| := (\sum_{j=1}^{d_c} \|\psi_j\|_{X^\sharp}^2)^{1/2}$ . Let  $\Omega$  be a neighborhood of 0 in  $X$  such that

$$\langle \Psi_c, \phi \rangle \in S_a, \quad \|\Pi^c \phi\|_X \leq a_0, \quad \text{and} \quad Q \leq b(a)$$

for  $\phi \in \Omega$ , where

$$Q := V(\langle \Psi_c, \phi \rangle) + \left( PC + \frac{\bar{K} K_\infty \|\Psi_c\| KC}{\beta_0 - \mu'} \right) (\|\Pi^s \phi\|_X + \|F_{*, \delta}(\Pi^c \phi)\|_X),$$

and consider the function  $W(\phi)$  on  $\Omega$  defined by

$$W(\phi) := V(\langle \Psi_c, \phi \rangle) + P\|\Pi^s \phi - F_{*, \delta}(\Pi^c \phi)\|_X, \quad \phi \in \Omega.$$

$W$  is continuous in  $\Omega$  with  $W(0) = 0$  and is positive in  $\Omega \setminus \{0\}$  because of (i) and (ii).

We will first certify the following claim.

**Claim 1.** *There exists a positive number  $c_0$  such that, for any  $t_0 \in \mathbb{R}^+$  and  $\phi \in X$  with  $W(\phi) \leq c_0$ , the solution  $x(t; t_0, \phi, f_\delta)$  exists on  $[t_0, t_0 + \tau_0]$  and satisfies  $x_t(t_0, \phi, f_\delta) \in \Omega$  for  $t \in [t_0, t_0 + \tau_0]$ ; in particular,  $\|\Pi^c x_t(t_0, \phi, f_\delta)\|_X \leq a_0$  in this interval.*

Indeed, suppose that  $x_t(t_0, \phi, f_\delta)$  is defined on the interval  $[t_0, t_0 + t_*)$  with  $t_* \leq \tau_0$ . Then

$$x_t(t_0, \phi, f_\delta) = T(t - t_0)\phi + \lim_{n \rightarrow \infty} \int_{t_0}^t T(t - s) \Gamma^n f_\delta(x_s(t_0, \phi, f_\delta)) ds$$

for  $t \in [t_0, t_0 + t_*]$ ; so

$$\|x_t(t_0, \phi, f_\delta)\|_X \leq M\|\phi\|_X + \int_{t_0}^t M\zeta_*(\delta)\|x_s(t_0, \phi, f_\delta)\|_X ds,$$

where  $M := \sup_{0 \leq t \leq \tau_0} \|T(t)\|_{\mathcal{L}(X)}$ . By Gronwall's inequality

$$\|x_t(t_0, \phi, f_\delta)\|_X \leq M\|\phi\|_X e^{M\zeta_*(\delta)(t-t_0)} \leq M\|\phi\|_X e^{M\zeta_*(\delta)\tau_0}, \quad t \in [t_0, t_0 + t_*],$$

which means that  $x_t(t_0, \phi, f_\delta)$  can be defined on the interval  $[t_0, t_0 + t_*]$  and therefore on  $[t_0, t_0 + \tau_0]$  (cf. [22, Corollary 1]). Thus it turns out that if  $\|\phi\|_X$  is small enough,  $x_t(t_0, \phi, f_\delta)$  exists on  $[t_0, t_0 + \tau_0]$  and moreover belongs to  $\Omega$  in this interval. The claim readily follows from the fact that  $\inf\{W(\phi) : \phi \in \Omega, \|\phi\|_X \geq r\} > 0$  for small  $r > 0$ , together with the property of  $\Omega$ .

Now given  $t_0 \in \mathbb{R}^+$  and  $\phi \in X$  with  $W(\phi) \leq c_0$ , consider the solution  $x(t) := x(t; t_0, \phi, f_\delta)$ . By Proposition 2 (i)

$$\|\Lambda_{*,\delta}(\Pi^c x_t)(s)\|_X \leq \|\Lambda_{*,\delta}(\Pi^c x_t)\|_{Y_\eta} e^{\eta|s|} \leq e^{\eta|s|} 2C \|\Pi^c x_t\|_X, \quad s \in \mathbb{R};$$

hence taking account of  $\Lambda_{*,\delta}(\Pi^c x_t)(s) = x_{t+s}(t, \hat{x}_t, f_\delta)$  for  $s \in \mathbb{R}$  (Proposition 3 (ii)), we get

$$\|x_{t+s}(t, \hat{x}_t, f_\delta)\|_X \leq e^{\eta\tau_0} 2C \|\Pi^c x_t\|_X, \quad s \in [-\tau_0, 0],$$

where  $\hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t)$ . Set  $y^\circ(t+s; t) := \langle\langle \Psi_c, x_{t+s}(t, \hat{x}_t, f_\delta) \rangle\rangle$ . Then

$$\begin{aligned} |y^\circ(t+s; t)| &\leq K_\infty \|\Psi_c\| \|x_{t+s}(t, \hat{x}_t, f_\delta)\|_X \\ &\leq K_\infty \|\Psi_c\| e^{\eta\tau_0} 2C \|\Pi^c x_t\|_X \\ &\leq K_\infty \|\Psi_c\| e^{\eta\tau_0} 2C a_0 \\ &= a/2, \quad s \in [-\tau_0, 0], \end{aligned}$$

hence  $y^\circ(s; t) \in S_{a/2}$  and thus  $V(y^\circ(s; t))$  is well-defined for  $s \in [t_0, t]$  with  $t \in [t_0, t_0 + \tau_0]$ .

We next confirm:

**Claim 2.**  $\sup\{W(x_t) : t \in [t_0, t_0 + \tau_0]\} \leq Q$  and  $W(x_{t_0+\tau_0}(t_0, \phi, f_\delta)) \leq c_0/2$ .

Indeed, fix a  $t \in [t_0, t_0 + \tau_0]$  and set  $z(s) := y^\circ(s; t)$  for  $s \in [t_0, t]$ . Since

$$y^\circ(s; t) = \langle\langle \Psi_c, x_s(t, \hat{x}_t, f_\delta) \rangle\rangle = \langle\langle \Psi_c, \Pi^c x_s(t, \hat{x}_t, f_\delta) \rangle\rangle, \quad s \in [t_0, t],$$

$z(s)$  is a solution of  $(CE_\delta)$  on  $[t_0, t]$  satisfying  $z(t) = y^\circ(t; t) = \langle\langle \Psi_c, \Pi^c x_t \rangle\rangle$ . By the property (i),  $\dot{V}(z(s)) \leq -V(z(s))$  for  $s \in [t_0, t]$ , which means

$$\frac{d}{ds} (e^{s-t} V(z(s))) = e^{s-t} (V(z(s)) + \dot{V}(z(s))) \leq 0,$$

so that

$$V(z(t)) - e^{t_0-t} V(z(t_0)) \leq \int_{t_0}^t \frac{d}{ds} (e^{s-t} V(z(s))) ds \leq 0;$$

consequently,

$$\begin{aligned}
V(\langle\langle \Psi_c, \Pi^c x_t \rangle\rangle) &\leq e^{t_0-t} V(y^\circ(t_0; t)) \\
&= e^{t_0-t} V(\langle\langle \Psi_c, \Pi^c x_{t_0} \rangle\rangle) + e^{t_0-t} (V(y^\circ(t_0; t)) - V(\langle\langle \Psi_c, \Pi^c x_{t_0} \rangle\rangle)) \\
&\leq e^{t_0-t} V(\langle\langle \Psi_c, \Pi^c x_{t_0} \rangle\rangle) + e^{t_0-t} \bar{K} |y^\circ(t_0; t) - \langle\langle \Psi_c, \Pi^c x_{t_0} \rangle\rangle| \\
&= e^{t_0-t} V(\langle\langle \Psi_c, \Pi^c \phi \rangle\rangle) + e^{t_0-t} \bar{K} |\langle\langle \Psi_c, \Pi^c x_{t_0}(t, \hat{x}_t, f_\delta) - \Pi^c x_{t_0} \rangle\rangle| \\
&\leq e^{t_0-t} V(\langle\langle \Psi_c, \Pi^c \phi \rangle\rangle) + e^{t_0-t} \bar{K} K_\infty \|\Psi_c\| \|\Pi^c x_{t_0}(t, \hat{x}_t, f_\delta) - \Pi^c x_{t_0}\|_X \\
&\leq e^{t_0-t} V(\langle\langle \Psi_c, \Pi^c \phi \rangle\rangle) + e^{t_0-t} \bar{K} K_\infty \|\Psi_c\| K \int_{t_0}^t e^{\mu'(\theta-t_0)} \|\xi(\theta)\|_X d\theta,
\end{aligned}$$

where the last inequality is due to Proposition 4 (ii). Therefore, applying Proposition 6,

$$\begin{aligned}
W(x_t) &= V(\langle\langle \Psi_c, \Pi^c x_t \rangle\rangle) + P \|\xi(t)\|_X \\
&\leq e^{t_0-t} V(\langle\langle \Psi_c, \Pi^c \phi \rangle\rangle) + e^{t_0-t} \bar{K} K_\infty \|\Psi_c\| K \int_{t_0}^t e^{\mu'(\theta-t_0)} (C \|\xi(t_0)\|_X e^{-\beta_0(\theta-t_0)}) d\theta \\
&\quad + PC \|\xi(t_0)\|_X e^{-\beta_0(t-t_0)} \\
(33) \quad &\leq e^{t_0-t} V(\langle\langle \Psi_c, \Pi^c \phi \rangle\rangle) + \frac{\bar{K} K_\infty K C \|\Psi_c\|}{\beta_0 - \mu'} \|\xi(t_0)\|_X e^{t_0-t} + PC \|\xi(t_0)\|_X e^{-\beta_0(t-t_0)}.
\end{aligned}$$

In particular,

$$\begin{aligned}
W(x_{t_0+\tau_0}) &\leq e^{-\tau_0} V(\langle\langle \Psi_c, \Pi^c \phi \rangle\rangle) + \frac{\bar{K} K_\infty K C \|\Psi_c\|}{\beta_0 - \mu'} \|\xi(t_0)\|_X e^{-\tau_0} + PC \|\xi(t_0)\|_X e^{-\beta_0 \tau_0} \\
&\leq (1/2) V(\langle\langle \Psi_c, \Pi^c \phi \rangle\rangle) + (1/4) P \|\xi(t_0)\|_X + (1/4) P \|\xi(t_0)\|_X \\
&= (1/2) W(x_{t_0}) = (1/2) W(\phi) \leq (1/2) c_0.
\end{aligned}$$

Since  $\|\xi(t_0)\|_X \leq \|\Pi^s \phi\|_X + \|F_{*,\delta}(\Pi^c \phi)\|_X$ , (33) implies also

$$\begin{aligned}
&\sup\{W(x_t) : t \in [t_0, t_0 + \tau_0]\} \\
&\leq V(\langle\langle \Psi_c, \Pi^c \phi \rangle\rangle) + \frac{\bar{K} K_\infty K C \|\Psi_c\|}{\beta_0 - \mu'} \|\xi(t_0)\|_X + PC \|\xi(t_0)\|_X \\
&\leq V(\langle\langle \Psi_c, \phi \rangle\rangle) + \left( PC + \frac{\bar{K} K_\infty \|\Psi_c\| K C}{\beta_0 - \mu'} \right) (\|\Pi^s \phi\|_X + \|F_{*,\delta}(\Pi^c \phi)\|_X) = Q,
\end{aligned}$$

as required.

By Claim 2, combined with Claim 1,  $x(t) = x(t; t_0, \phi, f_\delta)$  is defined on  $[t_0, t_0 + 2\tau_0]$ , and  $y^\circ(s; t) \in S_{a/2}$  still holds for  $s \in [t_0, t]$  with  $t \in [t_0, t_0 + 2\tau_0]$ .

Then we have:

**Claim 3.**  $\sup\{W(x_t) : t \in [t_0 + \tau_0, t_0 + 2\tau_0]\} \leq Q/2$  and  $W(x_{t_0+2\tau_0}) \leq c_0/2^2$ .

Indeed, let  $t \in [t_0 + \tau_0, t_0 + 2\tau_0]$ . By the same reasoning as in Claim 2 the inequality

$$\frac{d}{ds}(e^{s-t}V(y(s;t))) \leq 0, \quad s \in [t - \tau_0, t]$$

holds; so that  $V(z(t)) - e^{-\tau_0}V(z(t - \tau_0)) \leq 0$  and hence

$$\begin{aligned} V(\langle \Psi_c, \Pi^c x_t \rangle) &\leq e^{-\tau_0}V(\langle \Psi_c, \Pi^c x_{t-\tau_0} \rangle) + e^{-\tau_0} \bar{K} |\langle \Psi_c, \Pi^c x_{t-\tau_0}(t, \hat{x}_t, f_\delta) - \Pi^c x_{t-\tau_0} \rangle| \\ &\leq e^{-\tau_0}V(\langle \Psi_c, \Pi^c x_{t-\tau_0} \rangle) + e^{-\tau_0} \bar{K} K_\infty \|\Psi_c\| \|\Pi^c x_{t-\tau_0}(t, \hat{x}_t, f_\delta) - \Pi^c x_{t-\tau_0}\|_X \\ &\leq e^{-\tau_0}V(\langle \Psi_c, \Pi^c x_{t-\tau_0} \rangle) + e^{-\tau_0} \bar{K} K_\infty \|\Psi_c\| K \int_{t-\tau_0}^t e^{\mu'(\theta-t+\tau_0)} \|\xi(\theta)\|_X d\theta. \end{aligned}$$

Therefore, correspondingly to (33),

$$\begin{aligned} W(x_t) &\leq e^{-\tau_0}V(\langle \Psi_c, \Pi^c x_{t-\tau_0} \rangle) \\ &\quad + e^{-\tau_0} \bar{K} K_\infty \|\Psi_c\| K \int_{t-\tau_0}^t e^{\mu'(\theta-t+\tau_0)} (C \|\xi(t - \tau_0)\|_X e^{-\beta_0(\theta-t+\tau_0)}) d\theta \\ &\quad + PC \|\xi(t - \tau_0)\|_X e^{-\beta_0\tau_0} \\ &\leq e^{-\tau_0}V(\langle \Psi_c, \Pi^c x_{t-\tau_0} \rangle) + \frac{\bar{K} K_\infty K C \|\Psi_c\|}{\beta_0 - \mu'} \|\xi(t - \tau_0)\|_X e^{-\tau_0} \\ &\quad + PC \|\xi(t - \tau_0)\|_X e^{-\beta_0\tau_0} \\ &\leq (1/2)V(\langle \Psi_c, \Pi^c x_{t-\tau_0} \rangle) + (1/4)P \|\xi(t - \tau_0)\|_X + (1/4)P \|\xi(t - \tau_0)\|_X \\ &= (1/2)W(x_{t-\tau_0}) \\ &\leq (1/2) \sup\{W(x_\tau) : \tau \in [t_0, t_0 + \tau_0]\} \leq Q/2, \end{aligned}$$

where the last inequality follows from Claim 2. Letting  $t = t_0 + 2\tau_0$  in the above, we also see from claim 2 that

$$W(x_{t_0+2\tau_0}) \leq \frac{1}{2}W(x_{t_0+\tau_0}) \leq \frac{1}{2} \cdot \frac{c_0}{2} = \frac{c_0}{2^2},$$

and Claim 3 holds.

Repeating this argument, one can deduce in general that  $x(t) = x(t; t_0, \phi, f_\delta)$  is defined on  $[t_0, t_0 + n\tau_0]$ , and  $y^\circ(s; t) \in S_{a/2}$  holds for  $s \in [t_0, t]$  with  $t \in [t_0, t_0 + n\tau_0]$  for any  $n \in \mathbb{N}$ . Moreover,

$$\sup\{W(x_t) : t \in [t_0 + (n-1)\tau_0, t_0 + n\tau_0]\} \leq \frac{Q}{2^{n-1}} \quad \text{and} \quad W(x_{t_0+n\tau_0}) \leq \frac{c_0}{2^n}$$

for  $n \in \mathbb{N}$ . This means that  $x(t) = x(t; t_0, \phi, f_\delta)$  is actually defined on  $[t_0, \infty)$  and that

$$V(\langle \Psi_c, x_t(t_0, \phi, f_\delta) \rangle) + P \|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \leq Q 2^{-(t-t_0)/\tau_0}, \quad t \in [t_0, \infty).$$

In view of (i) and  $P > 1$ ,

$$b(|\langle \Psi_c, x_t(t_0, \phi, f_\delta) \rangle|) \leq Q 2^{-(t-t_0)/\tau_0} \leq b(a), \quad \|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \leq Q 2^{-(t-t_0)/\tau_0}.$$

Since

$$\|\Pi^c x_t(t_0, \phi, f_\delta)\|_X = \|\Phi_c \langle \Psi_c, x_t(t_0, \phi, f_\delta) \rangle\|_X \leq \|\Phi_c\| b^{-1} (Q 2^{-(t-t_0)/\tau_0})$$

with  $\|\Phi_c\| := (\sum_{j=1}^{d_c} \|\phi_j\|_X^2)^{1/2}$  and

$$\begin{aligned} \|\Pi^s x_t(t_0, \phi, f_\delta)\|_X &\leq \|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X + \|F_{*,\delta}(\Pi^c x_t)\|_X \\ &\leq Q 2^{-(t-t_0)/\tau_0} + L(\delta) \|\Pi^c x_t\|_X, \end{aligned}$$

so we obtain that for any  $\phi \in \Omega$  and  $t \in [t_0, \infty)$

$$\begin{aligned} \|x_t(t_0, \phi, f_\delta)\|_X &\leq \|\Pi^c x_t\|_X + \|\Pi^s x_t\|_X \\ &\leq Q 2^{-(t-t_0)/\tau_0} + (1 + L(\delta)) \|\Phi_c\| b^{-1} (Q 2^{-(t-t_0)/\tau_0}), \end{aligned}$$

which shows that the zero solution of  $(E_\delta)$  is uniformly asymptotically stable.  $\square$

Before concluding this section, we will provide an example to illustrate how our Theorem 6 is available for stability analysis of some concrete equations. Let us consider nonlinear (scalar) integral equation

$$(34) \quad x(t) = \nu \int_{-\infty}^t P(t-s)x(s)ds + f(x_t),$$

where  $\nu$  is a nonnegative real parameter,  $P$  is a nonnegative continuous function on  $\mathbb{R}^+$  satisfying  $\int_0^\infty P(t)dt = 1$  together with the condition  $\|P\|_{1,\rho} := \int_0^\infty P(t)e^{\rho t}dt < \infty$  and  $\|P\|_{\infty,\rho} := \text{ess sup}\{P(t)e^{\rho t} : t \geq 0\} < \infty$  for some positive constant  $\rho$ , and  $f \in C^1(X; \mathbb{C})$ ,  $X := L_\rho^1(\mathbb{R}^-; \mathbb{C})$ , satisfies  $f(0) = 0$  and  $Df(0) = 0$ . Eq. (34) is written as Eq. (E) with  $m = 1$  and  $K \equiv \nu P$ . The characteristic operator  $\Delta(\lambda)$  of Eq. (34) is given by  $\Delta(\lambda) = 1 - \nu \int_0^\infty P(t)e^{-\lambda t}dt$ . It is easy to see that if  $\nu > 1$ , then  $\Delta(\lambda_0) = 0$  for some positive  $\lambda_0$ ; hence  $\Sigma^u \neq \emptyset$ . Observe that  $|\Delta(\lambda)| \geq 1 - \nu$  if  $\text{Re } \lambda \geq 0$  and  $0 \leq \nu \leq 1$ . Thus, if  $0 \leq \nu < 1$ , then  $\Sigma^u \cup \Sigma^c = \emptyset$ . Hence, by virtue of **the principle of linearized stability** for integral equations (e.g., [7, Theorem 3.15]), we get the following:

**Proposition 8.** *Under the above conditions on Eq. (34), the following statements hold true;*

- (i) *if  $0 \leq \nu < 1$ , then the zero solution of Eq. (34) is exponentially stable (in  $L_\rho^1$ );*
- (ii) *if  $\nu > 1$ , then the zero solution of Eq. (34) is unstable (in  $L_\rho^1$ )*

In the remainder of this section, we will treat Eq. (34) in the critical case  $\nu = 1$ , and investigate stability property for the zero solution of Eq. (34) by applying Theorem 6. In case  $\nu = 1$ , we easily see that  $\Sigma^u = \emptyset$  and  $\Sigma^c = \{0\}$ . Indeed, in this case, 0 is a simple root of the equation  $\Delta(\lambda) = 0$ , and  $E^c$  is 1-dimensional space with a basis  $\{\phi_1\}$ ,  $\phi_1 \equiv 1$ , together with  $\{\psi_1\}$ ,  $\psi_1 \equiv 1$ , as the dual basis of  $\{\phi_1\}$ ; see [24] for details. The projection  $\Pi^c$  is given



by the formula  $\Pi^c \phi = \Phi_c \langle\langle \Psi_c, \phi \rangle\rangle$ ,  $\forall \phi \in X$ , and hence

$$\begin{aligned} \Pi^c \phi &= \phi_1 \langle\langle \psi_1, \phi \rangle\rangle = \phi_1 \left( \int_{-\infty}^0 \int_{\theta}^0 \psi_1(\xi - \theta) P(-\theta) \phi(\xi) d\xi d\theta \right) \\ &= \Phi_c \left( \int_{-\infty}^0 P(-\theta) \left( \int_{\theta}^0 \phi(\xi) d\xi \right) d\theta \right). \end{aligned}$$

Thus, for a solution  $x(t)$  of Eq. (34), the component  $z_c(t)$  of  $\Pi^c x_t$  with respect to  $\Phi_c$  is given by

$$z_c(t) = \int_{-\infty}^t \hat{P}(t-s)x(s)ds$$

with  $\hat{P}(t) := \int_t^\infty P(\tau)d\tau$ , because of

$$\begin{aligned} z_c(t) &= \int_{-\infty}^0 P(-\theta) \left( \int_{\theta}^0 x(t+\xi) d\xi \right) d\theta = \int_{-\infty}^0 P(-\theta) \left( \int_{t+\theta}^t x(s) ds \right) d\theta \\ &= \int_{-\infty}^t P(t-\tau) \left( \int_{\tau}^t x(s) ds \right) d\tau = \int_{-\infty}^t \left( \int_{t-s}^\infty P(w) dw \right) x(s) ds. \end{aligned}$$

Observe that  $z_c(t)$  satisfies the ordinary equation

$$\dot{z}_c(t) = \hat{P}(0)x(t) + \int_{-\infty}^t (-P(t-s))x(s)ds = x(t) - \int_{-\infty}^t P(t-s)x(s)ds,$$

that is,  $\dot{z}_c(t) = f(x_t) = f(\Phi_c z_c(t) + \Pi^s x_t)$ . In particular, if  $x$  is a solution of Eq. (34) satisfying  $x_t \in W_{\text{loc}}^c(r, \delta)$  on an interval  $J$ , then  $\Pi^s x_t = F_*(\Phi_c z_c(t))$  on  $J$ ; hence we get  $\dot{z}_c(t) = f(\Phi_c z_c(t) + F_*(\Phi_c z_c(t)))$  on  $J$ . This observation leads to that  $G_c = 0$  and  $H_c = 1$  in the central equation (CE); in fact, by noticing that  $\Sigma^c = \{0\}$  and  $H_c x = \lim_{n \rightarrow \infty} \langle\langle \psi_1, \Gamma^n x \rangle\rangle = x$ ,  $\forall x \in \mathbb{C}$ , one can also certify this fact. Let us assume that  $f$  is of the form

$$(35) \quad f(\phi) = \epsilon \left( \int_{-\infty}^0 \hat{P}(-\theta) \phi(\theta) d\theta \right)^3 + g(\phi), \quad \forall \phi \in X,$$

where  $\epsilon$  is a nonzero real number, and  $g \in C^1(X; \mathbb{C})$  satisfies  $|g(\phi)| = o(\|\phi\|_X^3)$  as  $\|\phi\|_X \rightarrow 0$  (here,  $o$  means Landau's notation "small oh"). Indeed, noting that  $\|\hat{P}\|_{\infty, \rho} \leq (1/\rho)\|P\|_{\infty, \rho}$  and  $\|\hat{P}\|_{1, \rho} \leq (1/\rho)\|P\|_{1, \rho}$ , we see that the function  $f$  given by (35) satisfies  $f \in C^1(X; \mathbb{C})$  and  $f(0) = Df(0) = 0$ . Since

$$\begin{aligned} f(\Phi_c z_c(t) + F_*(\Phi_c z_c(t))) &= f(x_t) = \epsilon \left( \int_{-\infty}^0 \hat{P}(-\theta) x_t(\theta) d\theta \right)^3 + g(x_t) \\ &= \epsilon \left( \int_{-\infty}^t \hat{P}(t-s)x(s)ds \right)^3 + g(x_t) \\ &= \epsilon(z_c(t))^3 + g(\Phi_c z_c(t) + F_*(\Phi_c z_c(t))), \end{aligned}$$

we get  $f(\Phi_\epsilon w + F_*(\Phi_\epsilon w)) = \epsilon w^3 + g(\Phi_\epsilon w + F_*(\Phi_\epsilon w)), \forall w \in \mathbb{C}$ . Thus, the central equation (CE) of Eq. (34) becomes to the equation

$$(36) \quad \dot{z} = \epsilon z^3 + g(\phi_1 z + F_*(\phi_1 z)).$$

Since  $F_*(\phi_1 z) = o(z)$  as  $z \rightarrow 0$  by Theorem 5-(i), it follows that  $\dot{z} = \epsilon z^3 + o(z^3)$  as  $z \rightarrow 0$ ; consequently, one can easily see that the zero solution of Eq. (36) is uniformly asymptotically stable (resp. unstable) if  $\epsilon < 0$  (resp.  $\epsilon > 0$ ). Therefore, by virtue of Theorem 6, we get the following result:

**Proposition 9.** *Let  $\nu = 1$  in Eq. (34), and assume that  $f$  is of the form (35) (with nonzero constant  $\epsilon$  and  $g(\phi) = o(\|\phi\|_X^3)$  as  $\|\phi\|_X \rightarrow 0$  with  $g \in C^1(X; \mathbb{C})$ ). Then*

- (i) *if  $\epsilon < 0$ , then the zero solution of Eq. (34) is uniformly asymptotically stable (in  $L_\rho^1$ );*
- (ii) *if  $\epsilon > 0$ , then the zero solution of Eq. (34) is unstable (in  $L_\rho^1$ ).*

#### 4. APPENDIX

In this appendix we will prove the  $C^1$ -smoothness of the center manifold  $W_\delta^c$  of the equilibrium point 0 of Eq.  $(E_\delta)$ , and give stable/unstable manifold theorems (of the equilibrium point 0) of Eq.  $(E)$ .

**4.1. Smoothness of the center manifold  $W_\delta^c$ .** For a Banach space  $U$  with norm  $\|\cdot\|_U$  and  $\lambda \geq 0$ , we define

$$BC^\lambda(\mathbb{R}; U) = \{y \in C(\mathbb{R}; U) : \sup_{t \in \mathbb{R}} \|y(t)\|_U e^{-\lambda|t|} < \infty\}.$$

$BC^\lambda(\mathbb{R}; U)$  is a Banach space normed with  $\|y\|_{BC^\lambda(\mathbb{R}; U)} := \sup_{t \in \mathbb{R}} \|y(t)\|_U e^{-\lambda|t|}$ . We use, for abbreviation, the notation  $\|\cdot\|_{\lambda, U}$  instead of  $\|\cdot\|_{BC^\lambda(\mathbb{R}; U)}$ . Evidently, if  $\lambda \leq \lambda'$ , there is an inclusion map  $BC^\lambda(\mathbb{R}; U) \hookrightarrow BC^{\lambda'}(\mathbb{R}; U)$  with

$$\|y\|_{\lambda', U} \leq \|y\|_{\lambda, U} \quad \text{for } y \in BC^\lambda(\mathbb{R}; U).$$

In what follows, for  $\nu \geq 0$  we denote the inclusion map from  $BC^\lambda(\mathbb{R}; U)$  to  $BC^{\lambda+\nu}(\mathbb{R}; U)$  by the same notation  $J_\nu$  for all  $\lambda \geq 0$  and any Banach space  $U$ . Clearly,  $J_\nu$  belongs to  $\mathcal{L}(BC^\lambda(\mathbb{R}; U); BC^{\lambda+\nu}(\mathbb{R}; U))$ .

By restricting the functions with values in an open subset  $\mathcal{O}$  of  $U$ , we also use the symbols  $BC^\lambda(\mathbb{R}; \mathcal{O})$ ,  $BC^{\lambda'}(\mathbb{R}; \mathcal{O})$  and so on to denote open subsets of the spaces  $BC^\lambda(\mathbb{R}; U)$ ,  $BC^{\lambda'}(\mathbb{R}; U)$  and so on, respectively.

Let  $\mathcal{O}$  be an open set of  $U$ , and consider a continuous map  $h : \mathcal{O} \rightarrow V$ . Then  $h$  induces a map  $\tilde{h}$  from  $C(\mathbb{R}; \mathcal{O})$  to  $C(\mathbb{R}; V)$  by letting

$$[\tilde{h}(y)](t) := h(y(t)) \quad \text{for } y \in C(\mathbb{R}; \mathcal{O}) \text{ and } t \in \mathbb{R}.$$

We recall that if  $\sup_{u \in \mathcal{O}} \|h(u)\|_V < \infty$ , the induced map  $\tilde{h}$  is continuous as a map from  $BC^\lambda(\mathbb{R}; \mathcal{O})$  to  $BC^{\lambda'}(\mathbb{R}; V)$  for  $\lambda' > 0$  (cf. [6, Appendix IV]).

**Proposition 10.**  $\Lambda_{*,\delta}$  is of class  $C^1$  as a map from  $E^c$  to  $Y_{\eta'}$  for  $\eta' \in (\eta, \alpha)$ .

*Proof.* Let  $\hat{\eta}$  be a positive number such that  $\eta < \hat{\eta} < \alpha$  and set

$$C_*(\lambda) := \zeta_*(\delta_1) C C_1 \left( \frac{1}{\lambda - \varepsilon} + \frac{2}{\alpha + \lambda} + \frac{2}{\alpha - \lambda} \right) \quad \text{for } \lambda \in [\eta, \hat{\eta}].$$

By taking  $\delta_1 > 0$  small if necessary, we may assume that

$$C_*(\lambda) < \frac{1}{2}, \quad \zeta_*(\delta) < 1 \quad \text{for } 0 < \delta \leq \delta_1 \quad \text{and} \quad \lambda \in [\eta, \hat{\eta}].$$

Now let  $\lambda \in [\eta, \hat{\eta}]$ ,  $\mu$  be a nonnegative number with  $\lambda + \mu \leq \hat{\eta}$  and  $v \in BC^\mu(\mathbb{R}; \mathcal{L}(X; \mathbb{C}^m))$ .

Consider a map  $\mathcal{H}_0(v) : Y_\lambda \rightarrow Y_{\lambda+\mu}$  defined by

$$\begin{aligned} [\mathcal{H}_0(v)w_0](t) &:= \lim_{n \rightarrow \infty} \int_0^t T^c(t-s) \Pi^c \Gamma^n v(s) w_0(s) ds \\ &\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s) \Pi^u \Gamma^n v(s) w_0(s) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_{-\infty}^t T^s(t-s) \Pi^s \Gamma^n v(s) w_0(s) ds, \quad t \in \mathbb{R} \end{aligned}$$

for  $w_0 \in Y_\lambda$ . By (13) it follows that

$$\begin{aligned} \|[\mathcal{H}_0(v)w_0](t)\|_X &\leq \left| \int_0^t C C_1 \zeta_*(\delta) e^{\varepsilon|t-s|} (\|v\|_{\mu, \mathcal{L}(X; \mathbb{C}^m)} e^{\mu|s|}) (\|w_0\|_{Y_\lambda} e^{\eta|s|}) ds \right| \\ &\quad + \int_t^\infty C C_1 \zeta_*(\delta) e^{\alpha(t-s)} (\|v\|_{\mu, \mathcal{L}(X; \mathbb{C}^m)} e^{\mu|s|}) (\|w_0\|_{Y_\lambda} e^{\eta|s|}) ds \\ &\quad + \int_{-\infty}^t C C_1 \zeta_*(\delta) e^{-\alpha(t-s)} (\|v\|_{\mu, \mathcal{L}(X; \mathbb{C}^m)} e^{\mu|s|}) (\|w_0\|_{Y_\lambda} e^{\eta|s|}) ds \\ &\leq C_*(\lambda + \mu) \|v\|_{\mu, \mathcal{L}(X; \mathbb{C}^m)} \|w_0\|_{Y_\lambda} e^{(\lambda+\mu)|t|} \end{aligned}$$

for any  $w_0 \in Y_\lambda$  and  $t \in \mathbb{R}$ ; hence

$$(37) \quad \|\mathcal{H}_0(v)w_0\|_{Y_{\lambda+\mu}} \leq \frac{1}{2} \|v\|_{\mu, \mathcal{L}(X; \mathbb{C}^m)} \|w_0\|_{Y_\lambda}, \quad w_0 \in Y_\lambda.$$

$\mathcal{H}_0(v)$  induces a bounded linear map from  $Y_\lambda^{(1)} = \mathcal{L}(E^c; Y_\lambda)$  to  $Y_{\lambda+\mu}^{(1)} = \mathcal{L}(E^c; Y_{\lambda+\mu})$ , denoted  $\mathcal{H}(v)$ , by

$$[[\mathcal{H}(v)w](t)]\phi := [\mathcal{H}_0(v)w\phi](t) \quad \text{for } w \in Y_\lambda^{(1)} \text{ and } \phi \in E^c,$$

where  $w\phi \in Y_\lambda$  is given by  $[w\phi](t) := w(t)\phi$  for  $t \in \mathbb{R}$ , and (37) yields the estimate

$$(38) \quad \|\mathcal{H}(v)w\|_{Y_{\lambda+\mu}^{(1)}} \leq \frac{1}{2} \|v\|_{\mu, \mathcal{L}(X; \mathbb{C}^m)} \|w\|_{Y_\lambda^{(1)}}, \quad w \in Y_\lambda^{(1)}.$$

By setting  $\lambda = \eta$ ,  $\mu = 0$  and  $v_*(\psi) := \widetilde{Df_\delta}(\Lambda_{*,\delta}(\psi))$  for  $\psi \in E^c$ , we can consider a linear equation in  $Y_\eta^{(1)}$

$$(39) \quad A_1 = T^c(\cdot) + \mathcal{H}(v_*(\psi))A_1, \quad \psi \in E^c$$

since  $T^c(\cdot)$  belongs to  $Y_\eta^{(1)}$ . Because  $\|\widetilde{Df_\delta}(\Lambda_{*,\delta}(\psi))\|_{0,\mathcal{L}(X;\mathbb{C}^m)} \leq \zeta_*(\delta)$  (cf. (6)), (38) implies

$$(40) \quad \|\mathcal{H}(v_*(\psi))\|_{\mathcal{L}(Y_\eta^{(1)})} \leq \frac{1}{2}\zeta_*(\delta) \leq \frac{1}{2},$$

and hence  $A_1 = A_1(\psi)$  is uniquely determined for each  $\psi \in E^c$  and is given by

$$(41) \quad A_1(\psi) = (I_{Y_\eta^{(1)}} - \mathcal{H}(v_*(\psi)))^{-1}T^c(\cdot) = \sum_{n=0}^{\infty} (\mathcal{H}(v_*(\psi)))^n T^c(\cdot).$$

Let us take any  $\eta' \in (\eta, \hat{\eta}]$ . We will verify that  $J_{\eta'-\eta}\Lambda_{*,\delta}(\psi)$  is differentiable and

$$(42) \quad D(J_{\eta'-\eta}\Lambda_{*,\delta}(\psi)) = J_{\eta'-\eta}A_1(\psi), \quad \psi \in E^c$$

holds. Notice that  $\mathcal{F}_\delta(\psi + h, \Lambda_{*,\delta}(\psi + h)) - \mathcal{F}_\delta(\psi, \Lambda_{*,\delta}(\psi + h)) = T^c(\cdot)h = A_1(\psi)h - \mathcal{H}_0(v_*(\psi))A_1(\psi)h$ ,  $\forall h \in E^c$ , by (14) and (39). We thus get

$$\begin{aligned} & \Lambda_{*,\delta}(\psi + h) - \Lambda_{*,\delta}(\psi) - A_1(\psi)h \\ &= \mathcal{F}_\delta(\psi + h, \Lambda_{*,\delta}(\psi + h)) - \mathcal{F}_\delta(\psi, \Lambda_{*,\delta}(\psi)) - A_1(\psi)h \\ &= \mathcal{F}_\delta(\psi, \Lambda_{*,\delta}(\psi + h)) - \mathcal{F}_\delta(\psi, \Lambda_{*,\delta}(\psi)) - \mathcal{H}_0(v_*(\psi))A_1(\psi)h \end{aligned}$$

for  $h \in E^c$ . Since

$$f_\delta(\Lambda_{*,\delta}(\psi + h)(s)) - f_\delta(\Lambda_{*,\delta}(\psi)(s)) = \int_0^1 Df_\delta(\Lambda_\sigma(\psi)(s))d\sigma (\Lambda_{*,\delta}(\psi + h)(s) - \Lambda_{*,\delta}(\psi)(s)),$$

where  $\Lambda_\sigma(\psi) := (1 - \sigma)\Lambda_{*,\delta}(\psi) + \sigma\Lambda_{*,\delta}(\psi + h)$  for  $\sigma \in [0, 1]$ , it follows that

$$\mathcal{F}_\delta(\psi, \Lambda_{*,\delta}(\psi + h)) - \mathcal{F}_\delta(\psi, \Lambda_{*,\delta}(\psi)) = \mathcal{H}_0(v_h(\psi))(\Lambda_{*,\delta}(\psi + h) - \Lambda_{*,\delta}(\psi)),$$

where  $v_h(\psi)$  is an element in  $BC(\mathbb{R}; \mathcal{L}(X; \mathbb{C}^m))$  defined by

$$[v_h(\psi)](s) := \int_0^1 Df_\delta(\Lambda_\sigma(\psi)(s))d\sigma \quad \text{for } s \in \mathbb{R}.$$

Hence

$$\begin{aligned} \Lambda_{*,\delta}(\psi + h) - \Lambda_{*,\delta}(\psi) - A_1(\psi)h &= \mathcal{H}_0(v_h(\psi) - v_*(\psi))(\Lambda_{*,\delta}(\psi + h) - \Lambda_{*,\delta}(\psi)) \\ &\quad + \mathcal{H}_0(v_*(\psi))(\Lambda_{*,\delta}(\psi + h) - \Lambda_{*,\delta}(\psi) - A_1(\psi)h), \end{aligned}$$

and, applying (37) and Proposition 2 (i), we get

$$\begin{aligned}
 & \|J_{\eta'-\eta}\Lambda_{*,\delta}(\psi+h) - J_{\eta'-\eta}\Lambda_{*,\delta}(\psi) - J_{\eta'-\eta}A_1(\psi)h\|_{Y_{\eta'}} \\
 & \leq (1/2)\|v_h(\psi) - v_*(\psi)\|_{\eta'-\eta, \mathcal{L}(X;\mathbb{C}^m)} \|\Lambda_{*,\delta}(\psi+h) - \Lambda_{*,\delta}(\psi)\|_{Y_\eta} \\
 & \quad + (1/2)\|v_*(\psi)\|_{0, \mathcal{L}(X;\mathbb{C}^m)} \|J_{\eta'-\eta}\Lambda_{*,\delta}(\psi+h) - J_{\eta'-\eta}\Lambda_{*,\delta}(\psi) - J_{\eta'-\eta}A_1(\psi)h\|_{Y_{\eta'}} \\
 & \leq C\|v_h(\psi) - v_*(\psi)\|_{\eta'-\eta, \mathcal{L}(X;\mathbb{C}^m)} \|h\|_X \\
 & \quad + (1/2) \|J_{\eta'-\eta}\Lambda_{*,\delta}(\psi+h) - J_{\eta'-\eta}\Lambda_{*,\delta}(\psi) - J_{\eta'-\eta}A_1(\psi)h\|_{Y_{\eta'}},
 \end{aligned}$$

so that

$$\begin{aligned}
 & \|J_{\eta'-\eta}\Lambda_{*,\delta}(\psi+h) - J_{\eta'-\eta}\Lambda_{*,\delta}(\psi) - J_{\eta'-\eta}A_1(\psi)h\|_{Y_{\eta'}} \\
 (43) \quad & \leq 2C\|v_h(\psi) - v_*(\psi)\|_{\eta'-\eta, \mathcal{L}(X;\mathbb{C}^m)} \|h\|_X.
 \end{aligned}$$

Notice that the continuous map  $Df_\delta : S_\delta \rightarrow \mathcal{L}(X;\mathbb{C}^m)$  satisfies  $\sup_{\phi \in S_\delta} \|Df_\delta(\phi)\|_{\mathcal{L}(X;\mathbb{C}^m)} \leq \zeta_*(\delta)$ , and hence by the fact stated in the preceding paragraph of the proposition, the induced map  $\widetilde{Df}_\delta : BC^\eta(\mathbb{R}; S_\delta) \rightarrow BC^{\eta'-\eta}(\mathbb{R}; \mathcal{L}(X;\mathbb{C}^m))$  is continuous because of  $\eta' - \eta > 0$ . Since  $v_h(\psi) - v_*(\psi) = \int_0^1 (\widetilde{Df}_\delta(\Lambda_\sigma(\psi)) - \widetilde{Df}_\delta(\Lambda_*(\psi)))d\sigma$  in  $BC^{\eta'-\eta}(\mathbb{R}; \mathcal{L}(X;\mathbb{C}^m))$ , the following inequality holds true:

$$\|v_h(\psi) - v_*(\psi)\|_{\eta'-\eta, \mathcal{L}(X;\mathbb{C}^m)} \leq \sup_{0 \leq \sigma \leq 1} \|\widetilde{Df}_\delta(\Lambda_\sigma(\psi)) - \widetilde{Df}_\delta(\Lambda_*(\psi))\|_{\eta'-\eta, \mathcal{L}(X;\mathbb{C}^m)}.$$

Hence  $\lim_{h \rightarrow 0} \|v_h(\psi) - v_*(\psi)\|_{\eta'-\eta, \mathcal{L}(X;\mathbb{C}^m)} = 0$  because of

$$\sup_{0 \leq \sigma \leq 1} \|\Lambda_\sigma(\psi) - \Lambda_*(\psi)\|_{Y_\eta} \leq \|\Lambda_{*,\delta}(\psi+h) - \Lambda_{*,\delta}(\psi)\|_{Y_\eta} \leq 2C\|h\|_X \rightarrow 0$$

as  $\|h\|_X \rightarrow 0$ . Thus (43) implies the differentiability of the map  $J_{\eta'-\eta}\Lambda_{*,\delta}(\psi)$  as well as  $D(J_{\eta'-\eta}\Lambda_{*,\delta}(\psi)) = J_{\eta'-\eta}A_1(\psi)$ .

We will finally certify the  $C^1$ -smoothness of the map  $J_{\eta'-\eta}\Lambda_{*,\delta}(\psi)$ . By (41)

$$J_{\eta'-\eta}A_1(\psi) = \sum_{n=0}^{\infty} J_{\eta'-\eta}(\mathcal{H}(v_*(\psi)))^n T^c(\cdot).$$

This series in  $Y_{\eta'}^{(1)}$  converges uniformly for  $\psi \in E^c$  because the norm of each term  $J_{\eta'-\eta}(\mathcal{H}(v_*(\psi)))^n$  does not exceed  $(1/2)^n$  by (40). So, by virtue of (42), it suffices to prove the continuity of the term  $J_{\eta'-\eta}(\mathcal{H}(v_*(\psi)))^n$  as a map from  $E^c$  to  $\mathcal{L}(Y_\eta^{(1)}, Y_{\eta'}^{(1)})$ . Given a positive integer  $n$ , put  $a_1 := (\eta' - \eta)/n$ . It is then easy to see that as a map from  $Y_\eta^{(1)}$  to  $Y_{\eta'}^{(1)}$

$$J_{\eta'-\eta}(\mathcal{H}(v_*(\psi)))^n = J_{a_1}^n(\mathcal{H}(v_*(\psi)))^n = (J_{a_1}\mathcal{H}(v_*(\psi)))^n = (\mathcal{H}(J_{a_1}v_*(\psi)))^n$$

holds for  $\psi \in E^c$ . Since  $\sup_{\phi \in S_\delta} \|Df_\delta(\phi)\|_{\mathcal{L}(X;\mathbb{C}^m)} \leq \zeta_*(\delta)$  again, the map  $J_{a_1}\widetilde{Df}_\delta : BC^\eta(\mathbb{R}; S_\delta) \rightarrow BC^{a_1}(\mathbb{R}; \mathcal{L}(X;\mathbb{C}^m))$  is continuous by the fact stated in the preceding paragraph of the proposition. Then  $J_{a_1}v_*(\psi)$  is continuous in  $\psi$ ; and so is  $J_{\eta'-\eta}(\mathcal{H}(v_*(\psi)))^n$ . This completes the proof.  $\square$

**Remark 1.** Let  $k$  be a positive integer. Then under the assumption that  $f$  is of class  $C^k$ , we can establish the  $C^k$ -smoothness of the center manifold  $W_\delta^c$ . In fact, the continuity of each term of the formal series, given by ( $m$  times) term-wise differentiation of (41) ( $m = 1, 2, \dots, k-1$ ), is guaranteed, together with the uniform convergence of the series, if we regard  $\Lambda_{*,\delta}$  as a map from  $E^c$  to  $Y_{\hat{\eta}}$  for a suitable  $\hat{\eta} \in (\eta, \alpha)$ ; for details see [27, Proposition 4].

**4.2. Stable manifold theorem and unstable manifold theorem.** In this subsection we will give (local) stable/unstable manifold theorems for the integral equation

$$(E) \quad x(t) = \int_{-\infty}^t K(t-s)x(s)ds + f(x_t)$$

under the assumption that the zero solution of (E) is hyperbolic.

For  $r > 0$  and  $\delta > 0$  we set

$$W_{\text{loc}}^s(r, \delta) = \left\{ \phi \in X : \|\Pi^s \phi\|_X < r, \ \|x_t(0, \phi, f)\|_X < \delta, \ t \in \mathbb{R}^+ \right\}.$$

Then  $W_{\text{loc}}^s(r, \delta)$  is called the local stable manifold (of the equilibrium point 0) of Eq.(E).

**Theorem 7.** *Assume that the zero solution of Eq.(E) is hyperbolic and that  $f \in C^k(X; \mathbb{C}^m)$  with  $f(0) = Df(0) = 0$ . Then there exist positive numbers  $r, \delta$ , and a  $C^k$ -map  $F^s : B_{E^s}(r) \rightarrow E^u$  with  $F^s(0) = 0$ , together with an open neighborhood  $\Omega_0$  of 0 in  $X$ , such that the following properties hold:*

- (i)  $W_{\text{loc}}^s(r, \delta) = \text{graph } F^s$ ; moreover,  $W_{\text{loc}}^s(r, \delta)$  is tangent to  $E^s$  at zero.
- (ii) To any  $\beta \in (0, \alpha)$  there corresponds a positive constant  $M$  such that

$$\|x_t(0, \phi, f)\|_X \leq M e^{-\beta t} \|\phi\|_X, \quad t \in \mathbb{R}^+, \quad \phi \in W_{\text{loc}}^s(r, \delta).$$

- (iii)  $W_{\text{loc}}^s(r, \delta)$  is locally positively invariant for Eq.(E), that is, if  $\phi \in W_{\text{loc}}^s(r, \delta)$ , we have  $x_\tau(0, \phi, f) \in W_{\text{loc}}^s(r, \delta)$  for  $\tau \in \mathbb{R}^+$  whenever  $\Pi^s x_\tau(0, \phi, f) \in B_{E^s}(r)$ .
- (iv) There exists a positive constant  $\beta_1$  with the property that if  $x(t)$  is a solution of Eq.(E) on an interval  $J = [t_0, t_1]$  satisfying  $x_t \in \Omega_0$  on  $J$ , then the inequality

$$\|\Pi^u x_t - F^s(\Pi^s x_t)\|_X \leq C \|\Pi^u x_{t_1} - F^s(\Pi^s x_{t_1})\|_X e^{\beta_1(t-t_1)}, \quad t \in J$$

holds true.

The properties (i) through (iii) of Theorem 7 can be proved in a similar manner to [27, Theorem 5]. Indeed, let  $\beta$  be a positive number less than  $\alpha$ , and  $Y_\beta^+$  the Banach space  $BC^\beta(\mathbb{R}^+; X)$ , that is,

$$Y_\beta^+ := BC^\beta(\mathbb{R}^+; X) = \left\{ y \in C(\mathbb{R}^+; X) : \sup_{t \in \mathbb{R}^+} \|y(t)\|_X e^{\beta t} < \infty \right\}$$

with norm  $\|y\|_{Y_\beta^+} := \sup_{t \in \mathbb{R}^+} \|y(t)\|_X e^{\beta t}$  for  $y \in Y_\beta^+$ . For sufficiently small  $r_0 > 0$  and  $\delta > 0$  one can define a map  $\mathcal{F} : B_{E^s}(r_0) \times B_{Y_\beta^+}(\delta) \rightarrow Y_\beta^+$  by

$$\begin{aligned} \mathcal{F}(\psi, y)(t) &:= y(t) - T^s(t)\psi - \lim_{n \rightarrow \infty} \int_0^t T^s(t-s) \Pi^s \Gamma^n f(y(s)) ds \\ &\quad + \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s) \Pi^u \Gamma^n f(y(s)) ds \end{aligned}$$

for  $(\psi, y) \in B_{E^s}(r_0) \times B_{Y_\beta^+}(\delta)$  and  $t \in \mathbb{R}^+$ . In view of the  $C^k$ -smoothness of the induced map  $\tilde{f} : B_{Y_\beta^+}(\delta) \rightarrow BC(\mathbb{R}^+; \mathbb{C}^m)$  (cf. [6, Appendix IV] and [27]), the map  $\mathcal{F}$  turns out, in contrast to  $\mathcal{F}_\delta$ , to be of class  $C^k$ . The implicit function theorem (e.g. see [18, Theorem 5.9 in Chapter I]) then yields the existence of the  $C^k$ -map  $\Lambda^s : B_{E^s}(r) \rightarrow B_{Y_\beta^+}(\delta)$  satisfying  $\mathcal{F}(\psi, \Lambda^s(\psi)) = 0$  for  $\psi \in B_{E^s}(r)$ ,  $r$  being some positive number with  $r \leq r_0$ .  $\Lambda^s(\cdot)$  plays a similar role to the one  $\Lambda_{*,\delta}(\cdot)$  does in the construction of center manifolds and, thus,  $F^s := \Pi^u \circ \text{ev}_0 \circ \Lambda^s$  is the desired one satisfying Properties (i), (ii) and (iii); we omit the details.

Property (iv), which is a result parallel to the result [5, Theorem 13.5.1] for ordinary differential equations, can be established by similar arguments to Propositions 1 through 6; so, we will give only a sketch of the proof. Given  $\psi \in E^s$ , the equation for  $y \in Y_\beta^+$

$$\begin{aligned} y(t) &= T^s(t)\psi + \lim_{n \rightarrow \infty} \int_0^t T^s(t-s) \Pi^s \Gamma^n f_\delta(y(s)) ds \\ &\quad - \lim_{n \rightarrow \infty} \int_t^\infty T^u(t-s) \Pi^u \Gamma^n f_\delta(y(s)) ds, \quad t \in \mathbb{R}^+, \end{aligned}$$

possesses a unique solution  $\Lambda_\delta^*(\psi)$ , where  $\delta \in (0, \delta_1]$ , and  $f_\delta$  is a function defined by  $f_\delta(\phi) := \chi(\|\Pi^u \phi\|_X / \delta) \chi(\|\Pi^s \phi\|_X / \delta) f(\phi)$ ,  $\phi \in X$ , and  $\delta_1$  is a number satisfying (13) (cf. Proposition 1). Considering a map  $F_\delta^* : E^s \rightarrow E^u$  defined by

$$F_\delta^*(\psi) = - \lim_{n \rightarrow \infty} \int_0^\infty T^u(-s) \Pi^u \Gamma^n f_\delta(\Lambda_\delta^*(\psi)(s)) ds, \quad \psi \in E^s$$

which is indeed an extension of  $F^s$ , one can verify that for any  $\tau \in \mathbb{R}$  and  $\hat{\phi} \in W_\delta^s := \{\psi + F_\delta^*(\psi) : \psi \in E^s\}$ , the solution  $x(t; \tau, \hat{\phi}, f_\delta)$  of  $(E_\delta)$  exists on  $[\tau, \infty)$ , and it satisfies the relation  $\Pi^u x_t(\tau, \hat{\phi}, f_\delta) = F_\delta^*(\Pi^s x_t(\tau, \hat{\phi}, f_\delta))$  for any  $t \geq \tau$  (cf. Proposition 3-(iii)). Let  $x$  be a solution of  $\text{Eq.}(E_\delta)$  on an interval  $J := [t_0, t_1]$ , and let  $\tau \in J$ . Then, putting  $\hat{x}_\tau := \Pi^s x_\tau + F_\delta^*(\Pi^s x_\tau)$  we get the following inequalities:

$$\|\Pi^s x_t - \Pi^s x_t(\tau, \hat{x}_\tau, f_\delta)\|_X \leq K \int_\tau^t e^{\mu(\sigma-t)} \|\Pi^u x_\sigma - \Pi^u x_\sigma(\tau, \hat{x}_\tau, f_\delta)\|_X d\sigma, \quad \tau \leq t \leq t_1;$$

$$\|\Pi^s x_t - \Pi^s x_t(\tau, \hat{x}_\tau, f_\delta)\|_X \leq K \int_\tau^t e^{\mu'(\sigma-t)} \|\Pi^u x_\sigma - F_\delta^*(\Pi^s x_\sigma)\|_X d\sigma, \quad \tau \leq t \leq t_1;$$

$$\|\Pi^s x_t(t_1, \hat{x}_{t_1}, f_\delta) - \Pi^s x_t(\tau, \hat{x}_\tau, f_\delta)\|_X \leq K \int_\tau^{t_1} e^{\mu'(\sigma-t)} \|\Pi^u x_\sigma - F_\delta^*(\Pi^s x_\sigma)\|_X d\sigma, \quad t \geq t_1;$$

here  $K := CC_1\zeta_*(\delta)$ ,  $\mu := \alpha - K$  and  $\mu' = \mu - KL(\delta)$  (cf. Propositions 4 and 5). Subsequently, utilizing these results and repeating the argument similar to Proposition 6 we can establish the inequality

$$\|\Pi^u x_t - F_\delta^*(\Pi^s x_t)\|_X \leq C\|\Pi^u x_{t_1} - F_\delta^*(\Pi^s x_{t_1})\|_X e^{\beta_1(t-t_1)}, \quad t \in J,$$

which implies Property (iv); here  $\beta_1$  is a (positive) number given by  $\beta_1 := \alpha\{1 - 2K/(2\alpha - K - KL(\delta))\}$ .

Similarly we can establish the existence of local unstable manifolds for  $\text{Eq.}(E)$ . For  $r > 0$  and  $\delta > 0$  consider the set

$$W_{\text{loc}}^u(r, \delta) = \{\phi \in X : \|\Pi^u \phi\|_X < r, \ \|x_t(0, \phi, f)\|_X < \delta, \ t \in \mathbb{R}^-\}.$$

Then we have the following theorem on the existence of  $C^k$ -smooth local unstable manifolds for  $\text{Eq.}(E)$ ; we omit the proof of the theorem.

**Theorem 8.** *Assume that the zero solution of  $(E)$  is hyperbolic and that  $f \in C^k(X; \mathbb{C}^m)$  with  $f(0) = Df(0) = 0$ . Then there exist positive numbers  $r$ ,  $\delta$ , and a  $C^k$ -map  $F^u : B_{E^u}(r) \rightarrow E^s$  with  $F^u(0) = 0$ , together with an open neighborhood  $\Omega_0$  of 0 in  $X$ , such that the following properties hold:*

- (i)  $W_{\text{loc}}^u(r, \delta) = \text{graph } F^u$ ; moreover,  $W_{\text{loc}}^u(r, \delta)$  is tangent to  $E^u$  at zero.
- (ii) To any  $\beta \in (0, \alpha)$  there corresponds a positive constant  $M$  such that

$$\|x_t(0, \phi, f)\|_X \leq M e^{\beta t} \|\phi\|_X, \quad t \in \mathbb{R}^-, \ \phi \in W_{\text{loc}}^u(r, \delta),$$

- (iii)  $W_{\text{loc}}^u(r, \delta)$  is locally negatively invariant for  $(E)$ , that is, if  $\phi \in W_{\text{loc}}^u(r, \delta)$ , we have  $x_\tau(0, \phi, f) \in W_{\text{loc}}^u(r, \delta)$  for  $\tau \in \mathbb{R}^-$  whenever  $\Pi^u x_\tau(0, \phi, f) \in B_{E^u}(r)$ .
- (iv) There exists a positive constant  $\beta_1$  with the property that if  $x(t)$  is a solution of  $\text{Eq.}(E)$  on an interval  $J = [t_0, t_1]$  satisfying  $x_t \in \Omega_0$  on  $J$ , then the inequality

$$\|\Pi^s x_t - F^u(\Pi^u x_t)\|_X \leq C\|\Pi^s x_{t_0} - F^u(\Pi^u x_{t_0})\|_X e^{-\beta_1(t-t_0)}, \quad t \in J$$

holds true.

**Remark 2.** We can also establish the existence and  $C^k$ -smoothness of center-stable/center-unstable manifolds for  $\text{Eq.}(E)$  provided that  $f$  is of class  $C^k$ . We will, however, omit the statements and the proofs of the theorems.

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